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Author
Dougherty, Michael Joseph

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The Geometry and Topology of the Dual Braid Complex

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by

Michael Joseph Dougherty

Committee in Charge:

Professor Jon McCammond, Chair

Professor Daryl Cooper

Professor Darren Long

June 2018

The Dissertation of
Michael Joseph Dougherty is approved:

Professor Daryl Cooper

Professor Darren Long

Professor Jon McCammond, Committee Chairperson

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The Geometry and Topology of the Dual Braid Complex

by

Michael Joseph Dougherty

To my mom

Acknowledgements

I could not have done this alone. I have attempted to use this space to thank some of those who have helped me along the way, but the list is seriously incomplete and these thanks alone are not nearly sufficient.

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Curriculum Vitæ

Michael Joseph Dougherty

Education

Ph.D. Mathematics | University of California, Santa Barbara | 2018 (exp)
M.A. Mathematics | University of California, Santa Barbara | 2014
B.A. Mathematics | Grinnell College | 2012

Publications

"Undesired Parking Spaces and Contractible Pieces of the Noncrossing Partition Link" (with J. McCammond), *Electronic Journal of Combinatorics*, Vol 25, Issue 1 (2018).

"A Decomposition of Parking Functions by Undesired Spaces" (with M. Bruce, M. Hlavacek, R. Kudo, I. Nicolas), *Electronic Journal of Combinatorics*, Vol 23, Issue 3 (2016).

"Hankel Transforms of Linear Combinations of Catalan Numbers" (with C. French, B. Saderholm, and W. Qian) *Journal of Integer Sequences*, Vol 14, Article 11.5.1 (2011).

Awards

Graduate Fellowship, National Physical Sciences Consortium | 2012-2018
Outstanding Teaching Assistant | University of California, Santa Barbara | 2017

Research Interests

Geometric Group Theory | Coxeter groups, Artin groups, CAT(0) groups
Topology | configuration spaces, mapping class groups
Combinatorics | noncrossing partitions, hyperplane arrangements

Abstract

The Geometry and Topology of the Dual Braid Complex

Michael Joseph Dougherty

The symmetric group is a classic example in group theory and combinatorics, with many applications in areas such as geometry and topology. While the closely related braid group also has frequent appearances in the same subjects and is generally well-understood, there are still many unresolved questions of a geometric nature. The focus of this dissertation is on the *dual braid complex*, a simplicial complex introduced by Tom Brady in 2001 with natural connections to the braid group. In particular, many subcomplexes and quotients of the dual braid complex have interesting properties, including applications to the "intrinsic geometry" for the braid group. At the root of these analyses is the theory of *noncrossing partitions*, a modern combinatorial theory with close connections to the symmetric group. In this dissertation, we survey the connections between these areas and prove new results on the geometry and topology of the dual braid complex.

Contents

1. Introduction

Among the central tenets of geometric group theory is the belief that a group can be understood via its actions on metric spaces. There is plenty of evidence to support this claim, and the past few decades of research have proved fruitful in the understanding of groups with natural geometric actions. Despite this, there are still many well-known groups for which the geometric picture is not yet complete.

The *n-strand braid group*, defined by Emil Artin in 1925 [Art25], has remained a popular topic of research for nearly a century. The elements of this group are often pictured as equivalence classes of diagrams containing strands which cross over one another, with the group operation given by concatenating and rescaling. The group is generated by diagrams in which two adjacent strands cross over one another, and there are two types of relations, corresponding to the cases when two pairs of strands are disjoint or overlap.

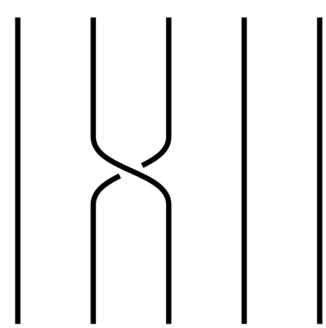

Figure 1.1: The generator σ_2 in the 5-strand braid group $\textsc{Braid}_5$

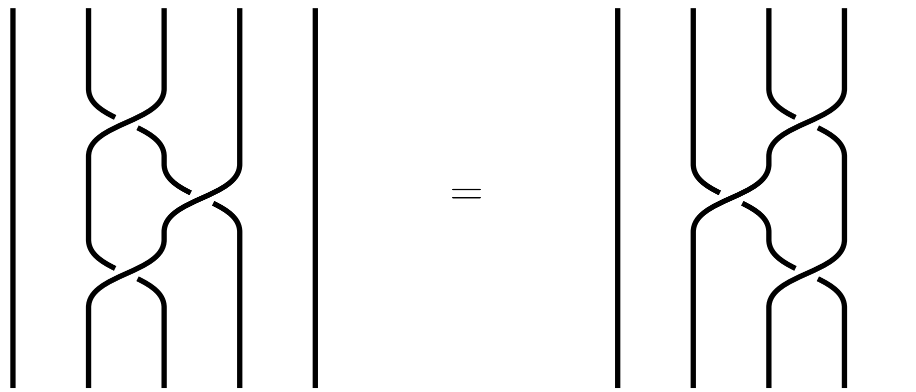

Figure 1.2: The relation $\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$ in the 5-strand braid group $\textsc{Braid}_5$

More concretely, let σ_i be represented by the diagram of n strands in which the i-th strand crosses over the $(i+1)$-st strand - see Figure 1.1. Then our two relations appear in Figures 1.2 and 1.3, giving us the following presentation:

$$\textsc{Braid}_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{cl} \sigma_i\sigma_j = \sigma_i\sigma_j & \text{if } |i-j| > 1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } |i-j| = 1 \end{array} \right\rangle$$

The braid groups are an exceptionally well-understood example in geometric group theory, with connections to fields such as knot theory, reflection groups, mapping class groups, and configuration spaces.

But on what types of metric spaces does the braid group act nicely? To a geometric group theorist, "nicely" means *geometrically*: properly discontinuously and cocompactly by isometries. These technical conditions ensure that a group action gives a geometric encoding to the essential features of the group. The existence of a geometric action of a group on a metric space can then be translated into interesting results for the group.

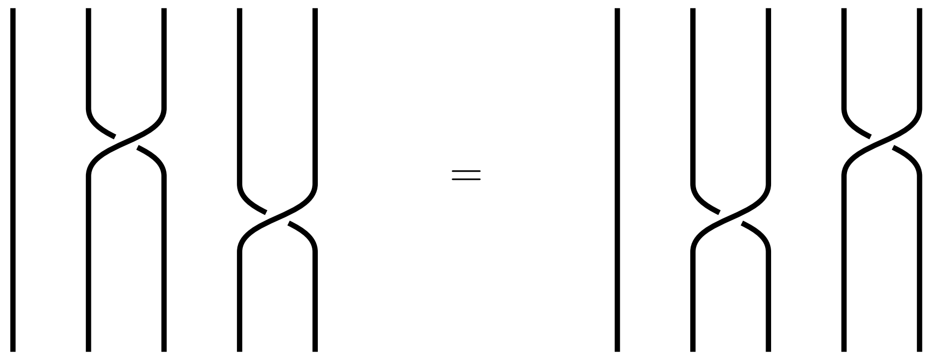

Figure 1.3: The relation $\sigma_2\sigma_4 = \sigma_4\sigma_2$ in the 5-strand braid group $\textsc{Braid}_5$

In the 1950s, Alexandrov explored the study of metric spaces with *nonpositive curvature* [BH99], and in 1987, Gromov highlighted Alexandrov's definition of a CAT(0) metric space. Groups with geometric actions on CAT(0) spaces have a wealth of properties, among them being a guaranteed solvable word problem. Many classes of groups were quickly proven to be CAT(0) after the concept was introduced, but the braid groups have resisted this classification so far.

The *Salvetti complex* is a classifying space for the braid group and is commonly used to understand the topological properties of the group. We may endow this space with a natural metric which then admits a geometric action by the braid group, but these complexes are not CAT(0) in general. Still, there is another option.

Aside from its diagrammatic depiction above, the braid group appears in many other contexts and may be described in several different ways. For example, the braid group is the fundamental group of a configuration space, meaning that we may view each braid as a motion of n distinct points in the disk. This is essentially just a change in perspective

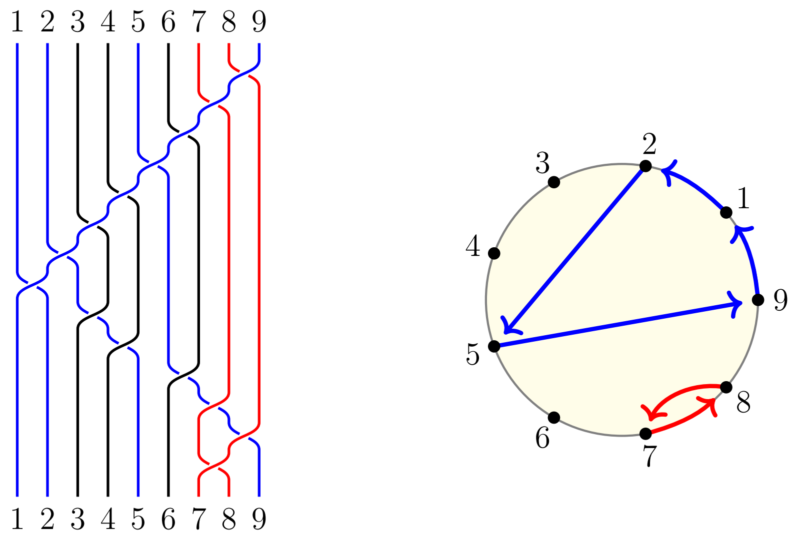

Figure 1.4: Two diagrams for the same braid

from the diagrams above, but this shift produces a powerful picture. See Figure 1.4 for an example of two ways to draw the same element of the braid group, where we read the diagram on the left from top to bottom.

This alternate perspective naturally yields a larger generating set for the braid group. If we choose n points in the disk by selecting the vertices of an inscribed regular n-gon, then any motion of these points with no collisions which returns them to their starting position (setwise) describes a braid. The simplest examples of these are found by partitioning the n vertices into smaller sets and then rotating each set counter-clockwise along the boundary of their convex hull. In the case of a two-element set, we thicken the convex hull (an edge) into a bigon before rotating - see Section 5.1 for details. Braids of this type are referred to as *dual simple braids* and correspond to the popular combinatorial objects known as *noncrossing partitions*. It is easy to convince oneself that these form a generating set D for the braid group. Now, take R to be the set of all relations $\delta_i \delta_j = \delta_k$ which hold among such elements in D. Then

$$\textsc{Braid}_n = \langle D \mid R \rangle$$

is the *dual presentation* for the braid group.

Just as the Salvetti complex is naturally associated to the usual presentation for the braid group, there is an analogous cell complex associated to the dual presentation. The *dual braid complex* $\mathcal{D}_n$, defined by Tom Brady in 2001 [Bra01], is a contractible simplicial complex which arises naturally from the presentation above. This complex was endowed with a metric by Brady with Jon McCammond in 2010 [BM10]; with this metric, the dual braid complex admits a geometric action by the braid group and is conjectured to be CAT(0). To date, the dual braid complex is known to be CAT(0) when the number of strands is $n \leq 6$ ([BM10], [HKS16]), but the question remains open in general.

In this dissertation, I present several new results on the structure of the dual braid complex, developed in collaboration with Jon McCammond and Stefan Witzel [DMW]. In particular, we introduce a new collection of subcomplexes in the dual braid complex and establish their curvature properties. The structure of these subcomplexes involves the articulation of a new type of configuration space for graphs and a new characterization of braids with *boundary-parallel strands.* These tools represent progress in our understanding of the curvature for the dual braid complex and may yet be used to prove that the braid groups are CAT(0).

Theorem A. *Let $Y_{n,k}$ be the subcomplex of the dual braid complex $\mathcal{D}_n$ determined by selecting k specified strands to remain boundary-parallel. Then $Y_{n,k}$ splits as the metric product $\mathcal{D}_{n-k} \times \sigma \times \mathbb{R}$, where σ is a $(k-1)$-dimensional Euclidean simplex. Consequently, $Y_{n,k}$ is* CAT(0) *if and only if the smaller dual braid complex $\mathcal{D}_{n-k}$ is* CAT(0)*.*

The dissertation is structured as follows. Chapter 2 gives an overview of the relationship between the symmetric group and the braid group, and the more general case of Coxeter groups and Artin groups. Chapter 3 contains a review of several essential topological and geometric tools. Chapter 4 applies some of these tools to survey the area of configuration spaces and introduces a new type of configuration space for graphs. Chapter 5 delves into the combinatorics of noncrossing partitions and reviews the definition of the dual presentation. The central object of study, the dual braid complex, is defined and explored in Chapter 6. We conclude with Chapter 7, presenting the results on braids with boundary-parallel strands and their associated complexes.

2. Coxeter Groups and Artin Groups

One of the essential components in understanding the braid group is to understand its relationship with the symmetric group. More generally, this connection is an example of the relationship between Artin groups and Coxeter groups, two classes of groups defined by certain types of presentations and characterized by useful geometric interpretations. While we refrain from using the general theory of Coxeter groups or Artin groups throughout the dissertation, we review it here for cultural background.

In this chapter, we begin with a thorough discussion on the symmetric group and the braid group before moving on to the background for Coxeter groups and Artin groups.

2.1 Braids and Permutations

The symmetric group is ubiquitous throughout mathematics, and the closely related braid group is nearly as well-known. In this section, we discuss some of their properties and establish our conventions for the remainder of the dissertation.

Definition 2.1.1 (Permutations)**.** Let n be a positive integer and let $[n]$ denote the set $\{1, 2, \ldots, n\}$. A *permutation* is a bijection between two copies of $[n]$, where we distinguish the two copies by left and right. The *cycle* notation $\sigma = (a_1, \ldots, a_k)$ is used to indicate that, under the permutation σ, a_i on the left corresponds to a_{i+1} on the right. Likewise, a_i on the right corresponds to a_{i-1} on the left. We denote these as $a_i \cdot \sigma = a_{i+1}$ and

$\sigma \cdot a_i = a_{i-1}$, respectively. Each permutation may then be written as a product of disjoint cycles. Permutations may be interpreted as functions, in which case we consider the right copy of $[n]$ to be the domain. Then multiplication of permutations is performed left-to-right, matching function composition. For example, $(1\ 2)(2\ 3) = (1\ 3\ 2)$. These conventions are chosen to match those of the programming package Sage.

Definition 2.1.2 (Symmetric group)**.** The *symmetric group* of rank n is the group of permutations of $[n]$ under composition. The symmetric group may be presented as

$$\text{SYM}_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \sigma_i^2 = e \text{ and } \begin{array}{ll} \sigma_i\sigma_j = \sigma_i\sigma_j & \text{if } |i-j| > 1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } |i-j| = 1 \end{array} \right\rangle$$

where σ_i is the transposition $(i\ \ i+1)$ and $e = (1)$ is the identity permutation. We denote the n-cycle $(1\ 2\ \cdots\ n)$ by σ_n. For each permutation $\sigma \in \text{SYM}_n$, define the *sign* $sgn(\sigma)$ to be 1 or -1 when σ can be written by an even or odd number of transpositions, respectively. Finally, the set of permutations which can be written as the product of an even number of transpositions forms an index-2 subgroup of SYM_n known as the *alternating group* ALT_n.

By slightly modifying the presentation above, we may define the braid group, which has historically been studied by representing elements via diagrams of crossing strands. However, it is useful to us to utilize a more topological definition.

Definition 2.1.3 (Complex Braid Arrangement)**.** The symmetric group SYM_n acts on $\mathbb{C}^n$ by permuting coordinates, where each transposition $(i\ j)$ acts as a reflection through the hyperplane $z_i - z_j = 0$ in $\mathbb{C}^n$. The collection of such hyperplanes is the *complex braid* *arrangement* $\mathcal{A}_n$ and the complement $\mathbb{C}^n - \mathcal{A}_n$ is a path-connected topological space on which the symmetric group acts freely. The *pure braid group* is the fundamental group of this complement:

$$\text{PBRAID}_n = \pi_1(\mathbb{C}^n - \mathcal{A}_n).$$

The *braid group* is the fundamental group of the quotient of the complement by the free action by the symmetric group:

$$\text{BRAID}_n = \pi_1((\mathbb{C}^n - \mathcal{A}_n)/\text{SYM}_n).$$

By the foundational work of Deligne [Del72] and Brieskorn-Saito [BS72], both spaces have a contractible universal cover and are hence classifying spaces.

Notice that the complement of the complex braid arrangement consists of all n-tuples $(z_1, \ldots, z_n) \in \mathbb{C}^n$ with distinct entries. Fix the vector $\mathbf{z} = (1, \ldots, i, i{+}1, \ldots, n)$ in $\mathbb{C}^n - \mathcal{A}_n$ and consider the path $f_i : [0,1] \to \mathbb{C}^n - \mathcal{A}_n$ defined by

$$f_i(t) = (1, \ldots, i-1, i + \frac{1}{2}(1 - e^{\pi i t}), i + 1 - \frac{1}{2}(1 - e^{\pi i t}), i + 2, \ldots, n).$$

Then $f_i(0) = \mathbf{z}$ and $f_i(1) = (1, \ldots, i+1, i, \ldots, n)$. Under the quotient by SYM_n, all permutations of the entries of $\mathbf{z}$ are identified and this point is labeled by the set $\{1, \ldots, n\}$. The image of each path f_i then has its endpoint identified by this quotient and thus each f_i corresponds to a loop in the quotient $(\mathbb{C}^n - \mathcal{A}_n)/\text{SYM}_n$. One can then show that these loops generate the fundamental group of this quotient, i.e. the braid group.

Definition 2.1.4 (Braid Group)**.** Let n be a positive integer. The *n-strand braid group* may be presented as

$$\textsc{Braid}_n = \left\langle \beta_1, \ldots, \beta_{n-1} \;\middle|\; \begin{array}{ll} \beta_i\beta_j = \beta_i\beta_j & \text{if } |i-j| > 1 \\ \beta_i\beta_j\beta_i = \beta_j\beta_i\beta_j & \text{if } |i-j| = 1 \end{array} \right\rangle$$

where each β_i is represented by the image of the path f_i described above, under the quotient by $\textsc{Sym}_n$. The loops in the quotient lift to the complex braid arrangement and the endpoints of the lifted path determine a permutation which is independent of lift. This gives a surjective map from $\textsc{Braid}_n$ to $\textsc{Sym}_n$, the kernel of which is known as the *pure n-strand braid group*.

Our goal is to improve our algebraic understanding of the braid group by enhancing our geometric understanding of one of its classifying spaces. To this end, we are interested in ways of understanding the braid group which exhibit as much symmetry as possible. While the definition above makes clear how the braid group originates from the symmetric group, there is a slight shift in perspective worth considering.

Definition 2.1.5 (Configuration Spaces)**.** The *ordered configuration space* of n points in the space X is the space $\textsc{Conf}_n(X)$ of n-tuples in X^n with distinct entries. The symmetric group then $\textsc{Sym}_n$ acts freely on $\textsc{Conf}_n(X)$ by permuting the entries - we define the *unordered configuration space* $\textsc{UConf}_n(X)$ by the quotient $\textsc{Conf}_n(X)/\textsc{Sym}_n$. Just as the elements for $\textsc{Conf}_n(X)$ are ordered n-tuples, we may identify the elements in the quotient $\textsc{UConf}_n(X)$ with unordered n-element sets.

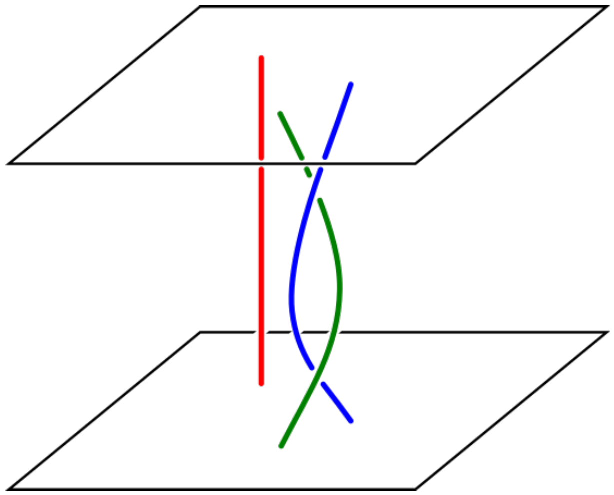

Figure 2.1: Strands

The braid arrangement complement $\mathbb{C}^n - \mathcal{A}_n$ is then the configuration space of n points in $\mathbb{C}$. Similar to how $\mathbb{C}$ can be deformation retracted to the unit disk $\mathbb{D}^2$, there is a deformation retraction of $\textsc{Conf}_n(\mathbb{C})$ to $\textsc{Conf}_n(\mathbb{D}^2)$, allowing us to consider elements of the braid groups as motions of n points in the disk. We leave the details of this procedure to Chapter 4.

It is useful to remember that the braid group can be viewed via each of these topological pictures: it is the fundamental group of both a configuration space and the complement of a complex hyperplane arrangement. The former example is important enough to warrant revisiting in Chapter 4.

Considering the braid group as the fundamental group of a configuration space gives us convenient pictures for each of its elements. If we choose a basepoint $\mathbf{z} = (z_1, \ldots, z_n)$ for $\textsc{Conf}_n(\mathbb{D}^2)$, then the set $Z = \{z_1, \ldots, z_n\}$ forms a basepoint in $\textsc{UConf}_n(\mathbb{D}^2)$. Any loop in the unordered configuration space which is based at Z then lifts to a path in $\textsc{Conf}_n(\mathbb{D}^2)$ which begins at $\mathbf{z}$ and ends at an n-tuple obtained by permuting the entries of $\mathbf{z}$.

Since $\textsc{Braid}_n = \pi_1(\text{U}\textsc{Conf}_n(\mathbb{D}^2), Z)$, we may view each braid by fixing a basepoint in $\textsc{Conf}_n(\mathbb{D}^2)$ and considering the paths which send this basepoint to a permutation of itself. Let $\gamma(t) : [0,1] \to \textsc{Conf}_n(\mathbb{D}^2)$ be such a path, where $\gamma(0) = \mathbf{z}$ and $\gamma(1)$ is a rearrangement of the entries of $\mathbf{z}$. Then if we write

$$\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$$

where each $\gamma_i(t)$ is a path in $\mathbb{D}^2$, we can draw n paths $(\gamma_i(t), t)$ in the cylinder $\mathbb{D}^2 \times [0,1]$ and observe that the paths do not intersect. We refer to each path in the cylinder as a *strand* and observe that isotopy classes of these sets of strands (which visually resemble what one intuitively considers a "braid") represent elements of $\textsc{Braid}_n$. The group operation corresponds to concatenating the cylinders representing two braids and rescaling. As before, we revisit this picture with greater attention to detail in Section 4.2.

As described above, each braid can be represented as a path which sends $\mathbf{z}$ to a permutation of that n-tuple, so there is a natural map $\textsc{Braid}_n \twoheadrightarrow \textsc{Sym}_n$ obtained by following strands. Since the pure braid group is the fundamental group of $\textsc{Conf}_n(\mathbb{D}^2)$, we see that $\text{P}\textsc{Braid}_n$ is the kernel of this map. In other words, we have the following short exact sequence:

$$\text{P}\textsc{Braid}_n \hookrightarrow \textsc{Braid}_n \twoheadrightarrow \textsc{Sym}_n$$

The connections between these three groups are essential to our understanding.

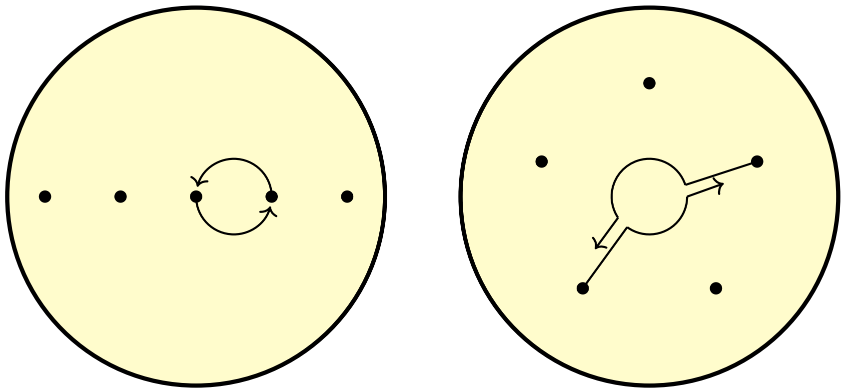

Figure 2.2: Two basepoints for the unordered configuration space of 5 points in $\mathbb{D}^2$ and two versions of positive half-twists

While there is no topological distinction between any two choices for the basepoint $\mathbf{z}$, different choices emphasize different types of symmetry for braids. Two natural choices are two arrange n distinct points in $\mathbb{D}^2$ in either a line or in a circle - see Figure 2.2. The former leads naturally to the usual diagrams for the braid group described in Chapter 1, while the latter exhibits a different type of symmetry.

Each picture described above suggests a set of generators for the braid group. When our basepoint consists of n vertices arranged in a line, a natural element of the fundamental group $\pi_1(\text{UC}\textsc{onf}_n(\mathbb{D}^2))$ is obtained by taking two adjacent vertices and rotating them counter-clockwise by π along the smallest circle which contains both. When our vertices are arranged in a circle which is centered at the origin, we can perform a similar motion by radially contracting a pair of vertices, rotating them counter-clockwise in a smaller concentric circle, and returning them to their swapped positions. These motions yield $n-1$ braids in the first picture and $\binom{n}{2}$ in the second, and both sets form generating sets for the braid group. We call these types of elements *positive half-twists*. See Section 4.2 for further details.

The two pictures for depicting braids given above correspond to two different perspectives on the braid group. Each of these corresponds to a distinct presentation for the braid group and each produces an associated cell complex which is homotopy equivalent to $\textsc{Conf}_n(\mathbb{D}^2)$. The focus of this work is on the setting corresponding to the choice of a circular basepoint.

2.2 Coxeter Groups

Coxeter groups are groups with exceptionally rich symmetry and good geometric properties. These groups are defined by presentations of a specific form, the prime example of which is the symmetric group. In this section, we review some of the basic notions regarding Coxeter groups and the geometries on which they act. Although we only use those which apply to the symmetric group in later chapters, the broader context is illuminating. For a deeper exploration of this material, see the standard references by Humphreys [Hum90] and Davis [Dav15].

Definition 2.2.1 (Coxeter Groups)**.** Let W be a group and let $S = \{s_1, \ldots, s_n\}$ be a subset of W. If, for each $i, j \in [n]$, there exists $m_{ij} \in \mathbb{Z}^+ \cup \{\infty\}$ such that $m_{ii} = 1$, $m_{ij} > 1$ when $i \neq j$, and $m_{ij} = m_{ji}$, then (W, S) is a *Coxeter system* if W is naturally isomorphic to the abstract group with presentation

$$W \cong \langle S \mid (s_i s_j)^{m_{ij}} = 1 \text{ for all } i, j \text{ with } m_{ij} < \infty \rangle.$$

In this case, we remark that $m_{ij} = 2$ corresponds to s_i and s_j commuting, and if $m_{ij} = \infty$, then $s_i s_j$ has infinite order. If (W, S) is a Coxeter system for some finite subset S of W, we say that W is a *Coxeter group*.

This information can be encoded in a simple graph with vertices labeled by the elements of S and an edge between the vertices labeled s_i and s_j whenever $m_{ij} \geq 3$. By convention, we label each edge with the number m_{ij} except when $m_{ij} = 3$. The resulting graph is called a *Coxeter diagram*, and the process is reversible - for each such graph Γ, there is a corresponding Coxeter group $\textsc{Cox}(\Gamma)$. We say that a Coxeter group is *irreducible* when its Coxeter diagram is connected. When the Coxeter diagram Γ is disconnected, then $\textsc{Cox}(\Gamma)$ is the direct product of irreducible Coxeter groups, one for each connected component of Γ.

We may also record this data in a matrix. Define the *Coxeter matrix* M to be the $n \times n$ matrix with (i, j) entry given by m_{ij}. Similarly, define the *Schläfli matrix* C to be the $n \times n$ matrix with (i, j) entry given by $-2\cos(\pi/m_{ij})$ when $m_{ij} \in \mathbb{Z}^+$ and -2 when $m_{ij} = \infty$.

Example 2.2.2 (Symmetric Group). The symmetric group $\textsc{Sym}_n$ is the best-known example of a spherical Coxeter group. To see that Definition 2.1.2 matches our format for Coxeter groups, notice that the generating set $S = \{\sigma_1, \ldots, \sigma_{n-1}\}$ for $\textsc{Sym}_n$ has relations which can be written as $\sigma_i\sigma_i = e$ for all i, $(\sigma_i\sigma_j)^2 = e$ when $|i - j| > 1$, and $(\sigma_1\sigma_j)^3 = e$ when $|i - j| = 1$. Then $\textsc{Sym}_n$ is indeed a Coxeter group - for example, $\textsc{Sym}_4$ with the presentation above has the following as its Coxeter matrix:

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

Similarly, the Schläfli matrix for this presentation is the following:

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

The associated Coxeter diagram is the connected graph with three vertices and two edges, depicted as A_3 in Figure 2.3.

In general, the symmetric group $\textsc{Sym}_n$ is $\textsc{Cox}(A_{n-1})$, where A_{n-1} is the connected graph on $n-1$ vertices with $n-2$ edges, i.e. a path. See Figure 2.3 for this and several other examples.

Before moving on to the associated geometry for Coxeter groups, it is worth discussing a special type of subgroup.

Definition 2.2.3 (Parabolic subgroups)**.** Let (W, S) be a Coxeter system. Then for each $I \subseteq S$, the subgroup W_I generated by I is referred to as the *parabolic subgroup* of W associated to I.

Parabolic subgroups of a Coxeter group satisfy several interesting properties. Proofs of the following facts may be found in [Hum90], Section 1.13.

Proposition 2.2.4. *If (W, S) is a Coxeter system and W_I is a parabolic subgroup of W, then W_I is a Coxeter group and (W_I, I) is a Coxeter system. Moreover, if Γ is the Coxeter diagram for (W, S) and $I \subseteq S$, then the full subgraph determined by the vertices labeled by elements in I is the Coxeter diagram for W_I.*

Proposition 2.2.5 (Intersections of Parabolics). *Let* (W, S) *be a Coxeter system and* $I, J \subseteq S$*. Then the intersection* $W_I \cap W_J$ *is the parabolic subgroup* $W_{I \cap J}$*.*

Each Coxeter group also has an associated action on a certain type of geometric object. Since each Coxeter group splits as a direct product of irreducible Coxeter groups, it suffices to discuss the irreducible case. As above, we simply outline what is known - the details may be found in [Hum90] and [Dav15].

Let (W, S) be a Coxeter system such that W is irreducible and let C be the associated Schläfli matrix. Then one can classify Coxeter groups by considering the eigenvalues of C. Notice that, since C is a real symmetric matrix, all of its eigenvalues are real. When the eigenvalues for C are all positive, the Coxeter group W acts geometrically (properly discontinuously and co-compactly by isometries) on a sphere in a Euclidean space. When C has zero as an eigenvalue with all others non-negative, W acts geometrically on a Euclidean space. In much of the literature, these are referred to as finite and affine Coxeter groups, respectively, but in this text we will refer to them by their intrinsic geometry: as *spherical* and *Euclidean Coxeter groups*.

To realize Coxeter groups as acting on a metric space, we give a formal definition for a familiar type of Euclidean transformation.

Remark 2.2.6 (Euclidean space). There are several appearances of real n-dimensional space throughout this dissertation, and we will distinguish them by the use of two different notations. Let $\mathbb{R}^n$ denote the real n-dimensional vector space with standard ordered basis vectors $\mathbf{e}_1 = (1, 0, \ldots, 0, 0), \ldots, \mathbf{e}_n = (0, 0, \ldots, 0, 1)$ and the ordinary dot product. On the other hand, define $\mathbb{E}^n$ to be n-dimensional Euclidean space with the standard Euclidean metric, with no specified origin or coordinate system.

Definition 2.2.7 (Reflections in $\mathbb{R}^n$)**.** Let $\alpha \in \mathbb{R}^n$ be nonzero and define H_α to be the orthogonal complement to α, i.e. the hyperplane of $\mathbb{R}^n$ given by

$$H_\alpha = \{v \in \mathbb{E} \mid v \cdot \alpha = 0\}.$$

Define the linear transformation $R_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ by $R_\alpha(v) = v - 2\frac{v\cdot\alpha}{\alpha\cdot\alpha}\alpha$. Then R_α sends α to $-\alpha$ and fixes H_α pointwise. Accordingly, R_α is the *reflection* through H_α and it is easy to see that ${R_\alpha}^2$ is the identity map for all $\alpha \in \mathbb{R}^n$.

Definition 2.2.8 (Root Systems)**.** Let Φ be a finite set of non-zero vectors in $\mathbb{R}^n$. Then Φ is a *root system* and elements of Φ are called *roots* if the following properties are satisfied:

1. If $\alpha \in \Phi$, then α and $-\alpha$ are the only multiples of α which appear in Φ.

2. R_α fixes Φ setwise for each $\alpha \in \Phi$.

If we further have that for each $\alpha, \beta \in \Phi$, $2\frac{\beta\cdot\alpha}{\alpha\cdot\alpha} \in \mathbb{Z}$, then we say that Φ is *crystallographic.*

Every spherical Coxeter group has a root system, and the group can be recovered from its root system. The spherical Coxeter groups that have crystallographic root systems are known as *Weyl groups*, and these are precisely the groups which arise in the classification of complex semisimple Lie algebras.

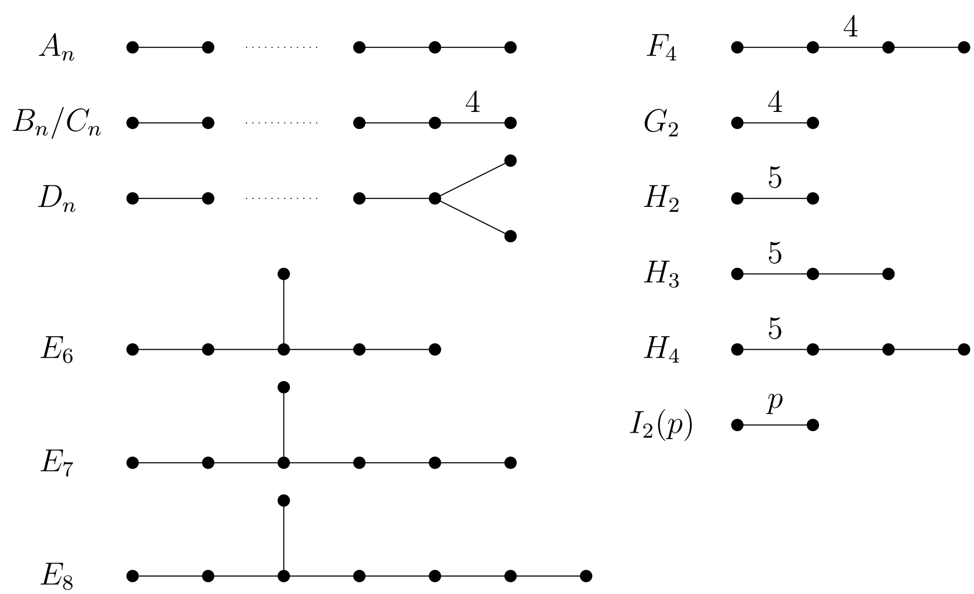

Figure 2.3: Coxeter diagrams for the spherical Coxeter groups

Definition 2.2.9 (Spherical Coxeter Groups)**.** If (W, S) is a Coxeter system where W is finite and irreducible and $S = \{s_1, \ldots, s_n\}$, then there is a collection of n vectors $\{\alpha_1, \ldots, \alpha_n\}$ in $\mathbb{R}_n$ which satisfy the following properties: first, for any $s_i, s_j \in S$, the vectors α_i and α_j have associated hyperplanes H_{α_i} and H_{α_j} which intersect with dihedral angle π/m_{ij}. Second, the reflections $R_{\alpha_1}, \ldots, R_{\alpha_n}$ generate a group which is isomorphic to W and the orbit of $\{\alpha_1, \ldots, \alpha_n\}$ under the action by this group is a root system Φ. The set of all reflections $\{R_\alpha \mid \alpha \in \Phi\}$ corresponds to the conjugacy class of S in W.

Finite irreducible Coxeter groups were classified by Coxeter in 1934 [Cox34] and are referred to as *irreducible spherical Coxeter groups* - see Figure 2.3 for a complete list of their Coxeter diagrams. As described above, each irreducible spherical Coxeter group has an associated root system Φ, and we may consider the collection of associated

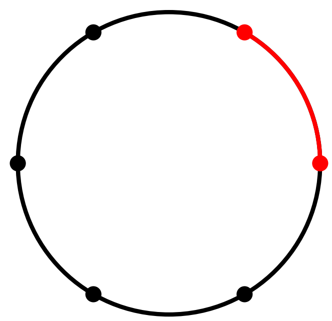

Figure 2.4: The A_2 Coxeter complex with one Coxeter shape highlighted

hyperplanes $\mathcal{H} = \{H_\alpha \mid \alpha \in \Phi\}$. The union of all hyperplanes in $\mathcal{H}$ forms a subset of $\mathbb{R}^n$ with a complement which consists of contractible connected components, each of which deformation retracts onto its intersection with the unit sphere $\mathbb{S}^{n-1}$. The closure of each component in $\mathbb{S}^{n-1}$ is referred to as a *chamber*, and the set of chambers determines a cell structure for the sphere which is invariant under the group action, called the *Coxeter complex* for W. Each maximal cell is a spherical simplex of a special isometry type, referred to as the *Coxeter shape* of type W.

Example 2.2.10 (Symmetric Group, reprise). Continuing from Example 2.2.2, we may exhibit the symmetric group $\textsc{Sym}_n$ as a spherical Coxeter group in the following manner. Let $S = \{\sigma_1, \ldots, \sigma_{n-1}\}$ and associate to each σ_i the vector $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ in $\mathbb{R}^n$. Then each hyperplane H_{α_i} is defined by the equation $x_i - x_{i+1} = 0$ and the corresponding reflection R_{α_i} acts on $\mathbb{R}^n$ by swapping the i-th and $(i+1)$-st entries in each vector. One can check that two hyperplanes H_{α_i} and H_{α_j} intersect with dihedral angle π/m_{ij} and the $n-1$ reflections $R_{\alpha_1}, \ldots, R_{\alpha_{n-1}}$ then generate a copy of $\textsc{Sym}_n$ - call this copy W. The orbit of $\{\alpha_1, \ldots, \alpha_{n-1}\}$ under the action by W is a collection of $2\binom{n}{2}$ unit vectors with corresponding hyperplanes defined by $x_i - x_j = 0$ for $i \neq j$. Just as each σ_i corresponds to the "adjacent transpositions" $(i\ i+1)$, these $\binom{n}{2}$ hyperplanes define reflections which correspond to all transpositions. Together, these $\binom{n}{2}$ hyperplanes form what is called the *real braid arrangement* and the complement of their union in $\mathbb{R}^n$ has $n!$ connected components, each determined by a linear ordering on the coordinates.

However, this situation differs from the definition above in that α_i lies in an n-dimensional real vector space, while $|S| = n - 1$. To reconcile this difference, notice that the $n-1$ hyperplanes $H_{\alpha_1}, \ldots, H_{\alpha_{n-1}}$ intersect in the line spanned by the vector $\mathbf{1} = (1, \ldots, 1)$; in fact, each α_i lies in the orthogonal complement $\mathbf{1}^{\perp}$, an isomorphic copy of $\mathbb{R}^{n-1}$ defined by the equation $x_1 + \cdots + x_n = 0$. Hence, we may restrict to this subspace and observe that the orbit of $\{\alpha_1, \ldots, \alpha_{n-1}\}$ under the action by W is a root system as described in Definition 2.2.9. Finally, the corresponding Coxeter complex is obtained by the intersection of the unit sphere $\mathbb{S}^{n-2}$ with the decomposition of $\mathbb{R}^{n-1}$ given by the hyperplanes above. The result is a tessellation on $\mathbb{S}^{n-2}$ into $n!$ spherical simplices of dimension $n-2$.

When $n = 3$, the real braid arrangement consists of 3 planes, given by the equations $x_1 - x_2 = 0$, $x_1 - x_3 = 0$, and $x_2 - x_3 = 0$ in $\mathbb{R}^3$. These three planes intersect in the line determined by the equations $x_1 = x_2 = x_3$, and the projection of their complement to the plane determined by the equation $x_1 + x_2 + x_3 = 0$ gives a decomposition of $\mathbb{R}^2$ into 6 connected components - see Figure 2.5. By radially contracting each connected component of the complement onto its intersection with the unit circle, we obtain the

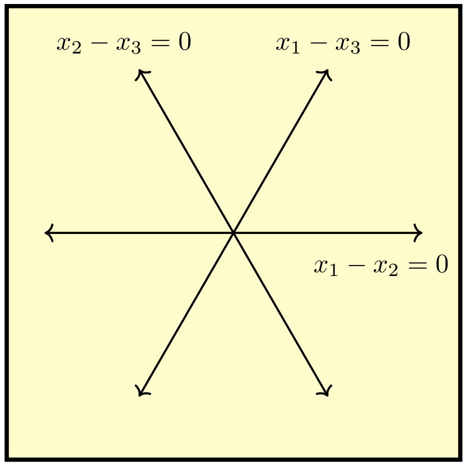

Figure 2.5: The real braid arrangement for A_2, projected onto the copy of $\mathbb{R}^2$ given by the equation $x_1 + x_2 + x_3 = 0$ in $\mathbb{R}^3$

Coxeter complex of type A_2, which consists of 6 spherical arcs of length $\frac{\pi}{3}$ and is depicted in Figure 2.4.

While our algebraic focus is on the spherical Coxeter group associated to A_n, the geometry of the Euclidean Coxeter groups will prove useful for us as well.

Definition 2.2.11 (Euclidean Coxeter Groups)**.** When a Coxeter group $\widetilde{W} = \textsc{Cox}(\Gamma)$ acts geometrically on a Euclidean space, we may obtain it from a spherical Coxeter group W by adding an affine hyperplane to the arrangement described in the remark above. This corresponds to adding one vertex to the Coxeter diagram for a spherical Coxeter group, and the ways that this can be done have been classified - see Figure 2.6. The result is that, instead of leaving a sphere invariant, this group induces a tessellation (i.e. a piecewise Euclidean cell structure) on a Euclidean space, which is also called the

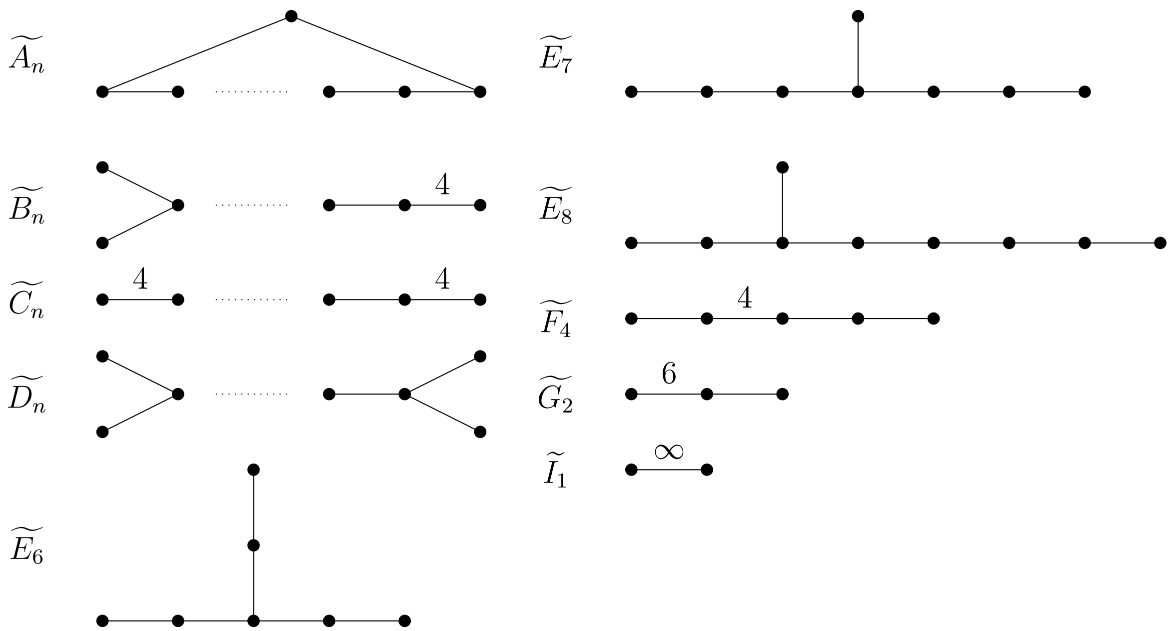

Figure 2.6: Coxeter diagrams for the Euclidean Coxeter groups

Coxeter complex of type $\widetilde{W}$. Each cell in this complex is a Euclidean simplex, referred to as the *Coxeter shape* of type $\widetilde{W}$.

Example 2.2.12 (Affine Symmetric Group). As before, let $S = \{\sigma_1, \ldots, \sigma_{n-1}\}$ be our generating set for the Coxeter group Sym_n, and for each σ_i let α_i be the associated root with corresponding hyperplane H_{α_i}. We may then extend this setting by including the *affine hyperplane* defined by the equation $x_1 - x_n = 1$ and its corresponding reflection via a natural generalization of Definition 2.2.7. Let τ represent this reflection and let $\widetilde{S} = S \cup \{\tau\}$. Then the group generated by $\widetilde{S}$ is the *affine symmetric group*; the result is infinite and induces a tessellation of $\mathbb{R}^n$ into simplices.

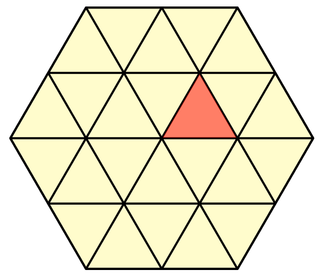

Figure 2.7: A portion of the $\widetilde{A}_2$ Coxeter complex with one Coxeter shape highlighted

The hyperplanes corresponding to the elements in $\widetilde{S}$ bound an infinite region R which intersects the hyperplane $x_1 + \ldots + x_n = 0$ in a Euclidean simplex. This simplex is an equilateral triangle when $n = 3$ (see Figure 2.7) and a tetrahedron with dihedral angles of $\pi/2$ and $\pi/3$ when $n = 4$. More specifically, this simplex is the $\widetilde{A}_{n-1}$ Coxeter shape and the region $R \subset \mathbb{R}^n$ can be written as the metric product of this simplex and $\mathbb{R}$. The geometry of this example is instrumental in later chapters.

2.3 Artin Groups

While the algebra and geometry of Coxeter groups is well understood, the closely related Artin groups remain relatively mysterious. Appearing in print for the first time in 1972 (Deligne [Del72] and Brieskorn-Saito [BS72]), Artin groups remain the subject of active research with many open problems. They are defined as follows.

Definition 2.3.1 (Artin Groups)**.** Let A be a group and let $S = \{s_1, \ldots, s_n\}$. For each $i, j \in [n]$, define $w_k(s_i, s_j)$ to be the word of length k which begins with s_i and strictly alternates between s_i and s_j. If, for each $i, j \in [n]$ with $i \neq j$, there exists $m_{ij} \in \mathbb{Z}^+ \cup \{\infty\}$

such that $m_{ii} = \infty$, $m_{ij} = m_{ij}$, and $m_{ij} > 1$, then A is an *Artin group* if A is naturally isomorphic to the group with presentation

$$A \cong \langle S \mid w_{m_{ij}}(s_i, s_j) = w_{m_{ji}}(s_j, s_i) \text{ for all } i \neq j \text{ with } m_{ij} < \infty \rangle.$$

In this case, we remark that m_{ij} corresponds to s_i and s_j commuting, and if $m_{ij} = \infty$, then there is no relation between s_i and s_j. Notice that if we impose additional relations of the form ${s_i}^2 = e$ for all $i \in [n]$, then the relation

$$w_{m_{ij}}(s_i, s_j) = w_{m_{ji}}(s_j, s_i)$$

can be rewritten as $(s_i s_j)^{m_{ij}}$. In other words, there is a natural Coxeter group associated to each Artin group (and vice-versa) obtained by sending generators in the natural way and imposing extra relations. We may then describe Artin groups by the same diagrams as their Coxeter counterparts: if $W = \textsc{Cox}(\Gamma)$ is a Coxeter group, then we refer to the corresponding Artin group as $\textsc{Art}(\Gamma)$. This group is said to be *spherical* or *Euclidean* when the associated Coxeter group $\textsc{Cox}(\Gamma)$ is spherical or Euclidean. In further analogy with the Coxeter setting, each subset $I \subseteq S$ determines a *parabolic subgroup* of $\textsc{Art}(\Gamma)$ defined to be the subgroup generated by I.

Just as the symmetric group $\textsc{Sym}_n = \textsc{Cox}(A_{n-1})$ is considered the quintessential spherical Coxeter group, the braid group $\textsc{Braid}_n = \textsc{Art}(A_{n-1})$ is the prime example of a spherical Artin group and a focal point for this dissertation.

Example 2.3.2 (Braid Groups). Let $n \leq 2$ be an integer. The Artin group of type A_{n-1} is the *n-strand braid group* $\text{BRAID}_n = \text{ART}(A_{n-1})$ and satisfies the presentation

$$\text{BRAID}_n = \left\langle \beta_1, \dots, \beta_{n-1} \;\middle|\; \begin{array}{ll} \beta_i\beta_j = \beta_i\beta_j & \text{if } |i-j| > 1 \\ \beta_i\beta_j\beta_i = \beta_j\beta_i\beta_j & \text{if } |i-j| = 1 \end{array} \right\rangle,$$

matching the one given in Definition 2.1.4.

The braid groups were defined by Emil Artin groups in 1925 [Art25], long before the more general Artin groups were defined, and they have made appearances in many different areas of mathematics, including geometric group theory, algebraic geometry, low-dimensional topology, and mathematical physics.

Similar to Proposition 2.2.5, the parabolic subgroups of an Artin group exhibit a useful structure, proven by Harm van der Lek in his dissertation [VdL83].

Proposition 2.3.3 (Parabolic Subgroups)**.** *Parabolic subgroups of Artin groups are also Artin groups. Furthermore, the intersection of the parabolic subgroups determined by generating subsets I and J is the parabolic subgroup determined by $I \cap J$.*

The connection between Coxeter groups and Artin groups provides the following definition.

Definition 2.3.4 (Pure Artin Groups)**.** Let Γ be a Coxeter diagram on vertex set S. Then the Artin group $\text{ART}(\Gamma)$ and the Coxeter group $\text{COX}(\Gamma)$ may both be defined by imposing relations on S which arise from Γ. Then there is a natural map from $\text{ART}(\Gamma)$ to $\text{COX}(\Gamma)$ obtained via the identity $S \to S$ on the generators. In other words, this map is the quotient of $\mathrm{Art}(\Gamma)$ by the subgroup generated by the squares of the generators in S. This map is clearly surjective, and the kernel is called the associated *pure Artin group*, denoted $\mathrm{PArt}(\Gamma)$. We then have the following short exact sequence:

$$\mathrm{PArt}(\Gamma) \hookrightarrow \mathrm{Art}(\Gamma) \twoheadrightarrow \mathrm{Cox}(\Gamma)$$

In the case of $\Gamma = A_{n-1}$, the pure Artin group is the pure braid group PBraid_n.

It is worth emphasizing that, unlike Coxeter groups, Artin groups are far less well-understood. Aside from a select few types of Artin groups, many basic group-theoretic questions remain open. A recurring theme in this dissertation is the question of whether Artin groups are CAT(0), but far simpler questions remain mysterious.

As described in recent work of Godelle and Paris [GP12], there are several conjectures on Artin groups which remain "wide open". The following four claims remain open for Artin groups in general, standing as a testament to how much work is left to do.

Conjecture 2.3.5 ([GP12])**.** *Let Γ be a Coxeter diagram.*

1. $\mathrm{Art}(\Gamma)$ *is torsion-free.*

2. *If Γ is irreducible and non-spherical, then* $\mathrm{Art}(\Gamma)$ *has trivial center.*

3. $\mathrm{Art}(\Gamma)$ *has solvable word problem.*

4. *The quotient of the* Salvetti complex *for* $\mathrm{Art}(\Gamma)$ *by* $\mathrm{Cox}(\Gamma)$ *is a classifying space for* $\mathrm{Art}(\Gamma)$*.*

It is worth noting that van der Lek's proof of Proposition 2.3.3 used only retractions and the general theory of fundamental groups to prove that smaller Artin groups inject as parabolic subgroups [VdL83]. Given the lack of a solution to the word problem for general Artin groups, van der Lek's work suggests that the topological perspective is an essential tool.

3. Simplices and Curvature

Many of the objects studied in this dissertation rely heavily on the relationship between combinatorial and geometric aspects of simplicial complexes. In this chapter we review the essential definitions and their consequences. Throughout this chapter, we follow many of the conventions and definitions used by Brady and McCammond [BM10].

3.1 Orders

In this chapter we provide several definitions for certain combinatorial relations and their connections with simplicial complexes. We begin by reviewing a common type of relation and one of its variants.

Definition 3.1.1 (Partial and Local Orders)**.** A relation $\leq$ on a set P is referred to as a *local order* if it satisfies the following properties for all $x, y \in P$:

1. *reflexivity* ($x \leq x$)
2. *anti-symmetry* (if $x \leq y$ and $y \leq x$, then $x = y$)

We write $x < y$ if $x \leq y$ and $x \neq y$, and we say that x is *covered* by y if $x < y$ and there does not exist z with the property that $x < z$ and $z < y$. A local order is promoted to a *partial order* if it satisfies the following additional criterion for all $x, y, z \in P$:

3 *transitivity* (if $x \leq y$ and $y \leq z$, then $x \leq z$)

When all three are satisfied, we refer to P as a *partially ordered set* or a *poset*.

Example 3.1.2 (Orders on $\mathbb{Z}^k$). The usual linear ordering on $\mathbb{Z}$ is a partial order which may then be extended to a partial order on $\mathbb{Z}^k$ by declaring that $(a_1, \ldots, a_k) \leq (b_1, \ldots, b_k)$ if $a_i \leq b_i$ for each $i \in [k]$. Alternatively, we can fix a positive integer n and define $x \leq_L y$ in $\mathbb{Z}$ if $x \leq y \leq x + n$. This may similarly be extended to a relation on $\mathbb{Z}^k$ and neither is transitive. Hence, this gives a local order on $\mathbb{Z}$ (and thus $\mathbb{Z}^k$) which is not a partial order.

While partial orders are far more common than their local counterparts, we make use of both, and thus we introduce them in parallel. In each of the above definitions, we may study collections which satisfy several consecutive relations.

Definition 3.1.3 (Chains)**.** Let L be a locally ordered set. A *k-chain* (or a *chain of length k*) in L is a set of $k+1$ totally ordered elements $l_0 < \cdots < l_k$, and any (proper) subset of these elements determines a *(proper) subchain* with the induced order. Such a chain is *saturated* if there is no longer chain between l_0 and l_k, and *maximal* if it is not a proper subchain of any other chain.

Before moving on to some associated topological spaces, we first name several useful tools for partially ordered sets.

Definition 3.1.4 (Bounded Poset)**.** Let P be a partially ordered set. Let $x \in P$ and $Q \subseteq P$. If $x \leq y$ for all $y \in Q$, then x is a *lower bound* for Q. Similarly, x is an *upper bound* if $y \leq x$ for all $y \in Q$. These need not be unique - see Figure 3.1 for an example. In the case that $Q = P$, lower and upper bounds are always unique if they exist, in which case we refer to them as *minimum* (denoted $\hat{0}$) and *maximum* (denoted $\hat{1}$) elements, respectively. Finally, if P has both a minimum and a maximum, then P is a *bounded* poset.

Definition 3.1.5 (Lattices)**.** Let P be a partially ordered set with $x, y \in P$. Then $z \in P$ is a *lower bound* for x and y if $z \leq x$ and $z \leq y$. If every other lower bound $w \in P$ for x and y satisfies $w \leq z$, then z is a *greatest lower bound* (or *meet*) for x and y. We may similarly define *upper bound* and *least upper bound* (or *join*) and note that while meets and joins do not always exist (see Figure 3.1), they are unique when they do. In these circumstances, we denote the meet and join by $x \wedge y$ and $x \vee y$ respectively. Finally, if every pair $x, y \in P$ has a meet and a join, we say that P is a *lattice*.

For any pair of elements in a locally ordered set, we may define the subset consisting of those which lie between them.

Definition 3.1.6 (Intervals)**.** Let L be a locally ordered set and $x, y \in L$. Then the bounded poset

$$[x, y] = \{z \in L \mid x \leq z \text{ and } z \leq y\}$$

is the *interval* between x and y. If each interval in L is finite, we say that L is *locally finite*. Finally, the Hasse diagram of an interval $[x, y]$ in L is obtained by taking the union of all paths from x to y in the Hasse diagram for L.

In a locally finite poset, every chain from x to y extends to a saturated chain in which adjacent elements form covering relations. In particular, the order in locally finite posets

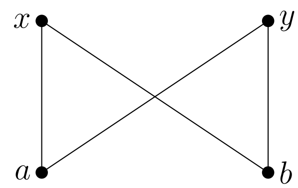

Figure 3.1: The Hasse diagram for a partially ordered set without well-defined meets and joins: a and b are both lower bounds for $\{x, y\}$, but neither $a \leq b$ nor $b \leq a$. Similarly, neither x nor y is the meet $a \wedge b$.

is entirely determined by its covering relations. When this is the case, these relations can be encoded in a directed graph.

Definition 3.1.7 (Hasse Diagram)**.** Let P be a locally finite partially ordered set. The *Hasse diagram* of P is a directed graph with vertices labeled by P. Let v_x and v_y be vertices labeled by $x, y \in P$. Then there is a directed edge from x to y if x is covered by y in P. More generally, there is a directed path from x to y in the Hasse diagram whenever $x \leq y$ in P. We typically draw the Hasse diagram without arrows on the edges, but orient the graph in the page so that each edge is oriented from bottom to top.

In certain cases, we can assign an integer to each element in a poset which measures the "rank" of an element.

Definition 3.1.8 (Rank Function)**.** Let L be a locally ordered set. A map $\rho : L \to \mathbb{Z}$ is a *rank function* if $x < y$ implies $\rho(x) < \rho(y)$ and whenever $x < y$ is a covering relation in L, we have $\rho(y) = \rho(x) + 1$.

Not all locally ordered sets admit a rank function. In the case of bounded posets, it is obviously necessary that all maximal chains in P have the same number of elements, and it is not difficult to show that this condition is sufficient.

Definition 3.1.9 (Grading)**.** Let L be a locally ordered set. Then L is *graded* if for all $x, y \in L$ with $x \leq y$, the interval $[x, y]$ has the property that all of its saturated chains from x to y have the same length. This common length is referred to as $\mathrm{rk}[x, y]$. If L has a unique minimum element $\hat{0}$ (as is always the case for intervals), we define the rank function $\mathrm{rk} : P \to \mathbb{Z}$ by $\mathrm{rk}(x) = \mathrm{rk}[\hat{0}, x]$ and observe that $\mathrm{rk}[x, y] = \mathrm{rk}(y) - \mathrm{rk}(x)$. When P is a bounded graded partially ordered set, its maximal chains must start at $\hat{0}$ and end at $\hat{1}$, and we refer to the length of its maximal chains as the *height* or *rank* of P.

Just as it is natural to consider automorphisms of a graph, there is a sensible way to describe maps which preserve the structure of a partial order.

Definition 3.1.10 (Order-preserving maps)**.** Let P and Q be partially ordered sets. A map $f : P \to Q$ is said to be *order-preserving* if $f(x) \leq f(y)$ in Q whenever $x \leq y$ in P. Similarly, f is *order-reversing* if $f(y) \leq f(x)$ in Q whenever $x \leq y$ in P. Order-preserving maps are *poset isomorphisms* if they are bijections with order-preserving inverses. A poset isomorphism is an *automorphism* when $P = Q$. Similarly, an order-reversing map $P \to Q$ is an *anti-isomorphism* (or *anti-automorphism* if $P = Q$) if it is a bijection with order-reversing inverse. When P and Q are locally finite, then order-preserving maps $P \to Q$ induce directed graph isomorphisms on the Hasse diagrams. The automorphisms of P then form a group under composition which we refer to as $\textsc{Aut}(P)$.

We illustrate several of these concepts with the following familiar example.

Example 3.1.11 (Boolean Lattice)*.* The *Boolean lattice of rank* n is the set of all subsets of $[n]$, partially ordered by inclusion and denoted BOOL_n. Covering relations in the Boolean lattice are those of the form $A < A \cup \{s\}$ - in other words, A is covered by the sets which contain A and one other element. We can see that BOOL_n has 2^n elements, bounded below by $\emptyset$ and above by $[n]$ itself. The $n!$ maximal chains of BOOL_n correspond to the permutations in SYM_n, since each permuted list gives a way to build $[n]$ one element at a time. Hence, each maximal chain has the same length and the grading on BOOL_n is given by cardinality: $\text{rk}(A) = |A|$. Additionally, BOOL_n is a lattice: for subsets A and B of $[n]$, their meet and join are $A \cap B$ and $A \cup B$ respectively. Each interval of BOOL_n is isomorphic to a smaller Boolean lattice. The order-preserving automorphisms of BOOL_n permute the one-element sets, and it is clear that each such permutation determines the entire map. In other words, the automorphism group of BOOL_n is isomorphic to the symmetric group SYM_n. The complement map, which sends A to $[n] - A$, is an example of an order-reversing automorphism.

3.2 Simplicial Complexes

In this section we review simplices and the complexes built out of them, both of which are common tools in combinatorial and geometric topology.

Definition 3.2.1 (Simplex)**.** A collection of $n + 1$ points in $\mathbb{R}^n$ is said to be in *general position* if there is no proper affine subspace which contains it. An *n-dimensional simplex*

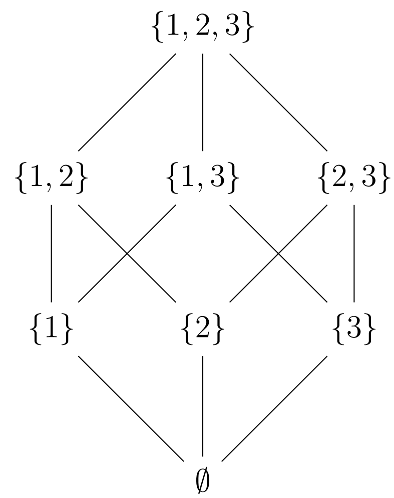

Figure 3.2: The Boolean lattice of rank 3

or n-*simplex* is the convex hull of $n+1$ points in general position in $\mathbb{R}^n$. A particularly useful example is given by the *standard* n-*simplex*: the convex subset of $\mathbb{R}^{n+1}$ described by

$$\{\mathbf{x} \in \mathbb{R}^{n+1} \mid x_1 + \cdots + x_{n+1} = 1 \text{ and } x_i \geq 0 \text{ for all } i \in [n+1]\},$$

i.e. the convex hull of the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}$ in $\mathbb{R}^{n+1}$. In general, the k-simplex on vertices $\mathbf{v}_1, \ldots, \mathbf{v}_{k+1}$ in general position in $\mathbb{R}^n$ may be written as the vectors

$$\{c_1\mathbf{v}_1 + \cdots + c_{k+1}\mathbf{v}_{k+1} \mid c_1 + \cdots + c_{k+1} = 1 \text{ and } c_i \geq 0 \text{ for all } i \in [k+1]\}.$$

We further say this simplex is *ordered* when there is a fixed linear ordering of its vertices. When this is the case, we refer to the $(k+1)$-tuple $(c_1, \ldots, c_{k+1})$ as the *barycentric coordinates* representing the point $c_1\mathbf{v}_1 + \cdots + c_{k+1}\mathbf{v}_{k+1}$ in the ordered simplex and observe that the coordinate vectors lie in the standard k-simplex in $\mathbb{R}^{k+1}$. Setting any of these coordinates equal to zero yields a *subsimplex* of codimension equal to the number of zeros. The subsimplices of a simplex are known as *faces*, and those of codimension 1 are *facets*.

Identifying the faces of a collection of simplices yields several types of topological spaces, each with useful structure.

Definition 3.2.2 (Simplicial Complex)**.** Let n be a nonnegative integer. Let X be a topological space obtained by gluing a collection of simplices of dimension at most n via iterative homeomorphic identification of subsimplices. Then X is a *simplicial complex* if every pair of simplices in X intersects in a subsimplex of each. As a consequence, any simplex in X is determined by its vertices. X is an *ordered simplicial complex* if it is built out of ordered simplices such that the linear orderings on each simplex may be combined into a local ordering on the vertex set for X.

Although we do not immediately need it, we take this opportunity to discuss a more general type of complex which is built out of simplices.

Definition 3.2.3 (Δ-complex)**.** Roughly speaking, a Δ-complex is a topological space obtained by taking a collection of ordered simplices and identifying certain subsimplices in a way which preserves the linear ordering on the vertex set for each ordered simplex. A Δ-complex is an ordered simplicial complex when each of its simplices is determined entirely by its endpoints. See Section 2.1 in [Hat02] for more details.

Definition 3.2.4 (Order Complex)**.** Let L be a locally finite locally ordered set. Then the *order complex* $\Delta(L)$ is the ordered simplicial complex with vertices labeled by elements

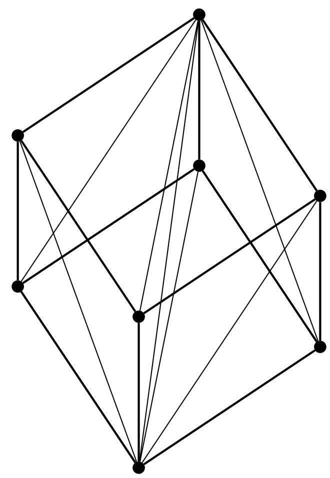

Figure 3.3: The order complex for the Boolean lattice of rank 3

of L and an ordered k-simplex on the vertices $\mathbf{v}_{\ell_0}, \dots, \mathbf{v}_{\ell_k}$ in $\Delta(L)$ whenever $\ell_0 < \cdots < \ell_k$ is a k-chain in L.

Example 3.2.5 (Boolean Lattice). There is a natural bijection between $\textsc{Bool}_n$ and the vertices of the unit cube $[0,1]^n$ in $\mathbb{R}^n$ given by sending $C \in \textsc{Bool}_n$ to the n-tuple $(c_1, \dots, c_n)$, where c_i is 1 if $i \in C$ and 0 if not. Then C is covered by D if and only if the vertex corresponding to D can be obtained from that of C by adding 1 to one of the coordinates. In other words, covering relations in $\Delta(\textsc{Bool}_n)$ correspond to the unit length edges in the unit cube, and maximal chains can be written as directed paths from $(0, \dots, 0)$ to $(1, \dots, 1)$ in the integer lattice. More generally, the convex hulls of these $n!$ paths in $\mathbb{R}^n$ are simplices which share a common edge from $(0, \dots, 0)$ to $(1, \dots, 1)$. We have observed that the order complex $\Delta(\textsc{Bool}_n)$ is a simplicial subdivision of a unit cube.

A more general version of this example can be obtained by studying the local order on $\mathbb{Z}^n$.

Example 3.2.6 (Integer Lattice). Consider the local order L_n on $\mathbb{Z}^n$ given by declaring $\mathbf{u} \leq \mathbf{v}$ if there exists $\mathbf{z} \in \mathbb{Z}^n$ such that $\mathbf{u} + \mathbf{z} = \mathbf{v}$ and every entry in $\mathbf{z}$ is either 0 or 1. Then for each $\mathbf{u} \in \mathbb{Z}^n$, the set of vertices of the form $\mathbf{u} + \mathbf{z}$ with the same restrictions gives a copy of the Boolean lattice within this local order. The order complex of L_n is then naturally identified with the tiling of Euclidean n-space by unit n-cubes, each of which has been subdivided into $n!$ simplices. In particular, each cube can be viewed as the order complex for the Boolean lattice.

The simplicial metrics appearing in these examples share a common isometry type among their simplices. In fact, there is often a natural metric associated to an order complex, which we explore in Section 3.5.

Among the benefits of defining these types of simplicial complexes is the existence of nice products - there is a canonical simplicial structure on the product of ordered simplicial complexes.

Example 3.2.7 (Products of Simplices). An ordered k-simplex is the order complex of the totally ordered vertex set $x_0 < \cdots < x_k$, so ordered simplices are the easiest examples of ordered simplicial complexes. Let σ and τ be ordered simplices on ordered vertex sets

$$u_0 <_\sigma \cdots <_\sigma u_l$$

and

$$v_0 <_\tau \cdots <_\tau v_m.$$

We can construct a simplicial structure for the product $\sigma \times \tau$ by endowing the vertex set $V = \{(u_i, v_j)\}$ with the product order $(u, v) \leq (u', v')$ if and only if $u \leq_\sigma u'$ and $v \leq_\tau v'$. Then $\Delta(V)$ is an ordered simplicial complex and the projections $\Delta(V) \to \sigma$ and $\Delta(V) \to \tau$ are simplicial maps. This leads to a map $\Delta(V) \to \sigma \times \tau$ which we can verify to be a homeomorphism. The proof of this claim can be found in Lemma 8.9 of [ES52].

Example 3.2.8. An ordered 1-simplex is simply an interval with a designated orientation, and we may see that it is the order complex of a totally ordered poset with 2 elements, i.e. the Boolean lattice $\textsc{Bool}_2$. A product of n 1-simplices is homeomorphic to an n cube and the vertices are given by the product of n copies of $\textsc{Bool}_2$, i.e. $\textsc{Bool}_n$. The order complex $\Delta(\textsc{Bool}_n)$ is homeomorphic to product of n 1-simplices with a simplicial structure which matches the simplicial structure described in Example 3.2.7.

Another example of this construction may be found in Hatcher's *prism operator*, which decomposes the product of an ordered n-simplex and an interval into $n + 1$ simplices [Hat02].

Understanding products of simplices is essentially enough to understand the general case for products of simplicial complexes.

Definition 3.2.9 (Ordered Simplicial Product)**.** Let Δ_1 and Δ_2 be ordered simplicial complexes on vertex sets $V(\Delta_1)$ and $V(\Delta_2)$ with local orders $\leq_1$ and $\leq_2$, respectively. Define the product order $\leq$ on the vertex set $V = V(\Delta_1) \times V(\Delta_2)$ by $(u, v) \leq (u', v')$ if and only if $u \leq_\sigma u'$ and $v \leq_\tau v'$. Then the order complex $\Delta(V)$ is the *ordered simplicial product* of Δ_1 and Δ_2, and we denote this ordered simplicial complex by $\Delta_1 \boxslash \Delta_2$. More generally, if Δ_1 and Δ_2 are Δ-complexes, then the product has a natural simplicial structure obtained in a similar fashion, by simply decomposing each product of simplices as described in Example 3.2.7. We also denote this product by $\Delta_1 \boxslash \Delta_2$ and observe that this aligns with the definition for ordered simplicial complexes.

The proposition below is proven in [ES52] for ordered simplicial complexes, albeit with different language and notation.

Proposition 3.2.10. *Let Δ_1 and Δ_2 be Δ-complexes. Then the ordered simplicial product $\Delta_1 \boxslash \Delta_2$ is homeomorphic to the direct product $\Delta_1 \times \Delta_2$.*

3.3 Polytopes and Links

While the order complex of a local order carries much of the same combinatorial data as its corresponding ordered set, it is often uninteresting from a topological perspective. For example, if a poset is bounded above or below, then its order complex is a topological cone which can be contracted to a bounding vertex. As this is the case for some of our intended applications, we will need a more topologically robust space.

Definition 3.3.1 (Cell Complexes)**.** Roughly speaking, a *cell complex* (or *CW complex*) is a topological space X which may be iteratively constructed, in which the k-th step consists of gluing closed balls of dimension k via attaching maps which identify their boundaries to the $(k-1)$-skeleton $X^{(k-1)}$. A cell complex is *regular* if the domain of each attaching map $\varphi : \partial\mathbb{D}^k \to X^{(k-1)}$ (i.e. a $(k-1)$-dimensional sphere) may be given a cell structure such that the restriction of φ to any cell in this structure is a homeomorphism onto a cell in $X^{(k-1)}$.

The simplicial complexes described in Section 3.2 are examples of cell complexes, where each cell is a simplex and identifications respect the simplicial structure. A particularly nice type of cell complex is obtained as the convex hull of a generic collection of points.

Definition 3.3.2 (Euclidean Polytopes)**.** A *Euclidean polytope* is the convex hull of a finite collection of points in general position in $\mathbb{E}^n$, referred to as the *vertices* of the polytope. Alternatively, such a polytope may be described as the intersection of a finite collection of half-spaces in $\mathbb{E}^n$. A Euclidean polytope can be given a natural cell structure: if S is a subset of the vertices which lies in the boundary of a proper half-space which contains the polytope, then the convex hull of S is a *proper face*. Each proper face is then also a Euclidean polytope. Together with the entire polytope and the empty face $\emptyset$, these form the set of *faces*. The faces of dimension $n-1$ are referred to as *facets*.

There is a spherical analogue for these polytopes.

Definition 3.3.3 (Spherical Polytopes)**.** A *spherical polytope* is the convex hull of a finite set of points in the n-sphere $\mathbb{S}^n$ which lies in an open hemisphere. Alternatively, a spherical polytope is the intersection of a finite collection of hemispheres in $\mathbb{S}^n$ with the same restriction that the intersection lies in an open hemisphere. If S is a subset of the defining vertices which lies in the boundary of a hemisphere containing the polytope, then the convex hull of S is a *proper face*, and we remark that each proper face is also a spherical polytope. With the empty face $\emptyset$ and the entire polytope, these form the set of *faces*. The faces of dimension $n-1$ are *facets*.

Hemispheres in the second definition play the role of half-spaces in the first; the relationship between these two allows us to define many similar objects in both contexts. In particular, we are interested in the metric spaces which can be built out of these types of polytopes.

Definition 3.3.4 (Polytopal Cell Complexes)**.** A cell complex is *polytopal* if its cells are each given by polytopes. A polytopal cell complex X is *regular* if each of its attaching maps identifies the boundary of a k-dimensional polytope P to the $k-1$ skeleton of X by homeomorphisms on each proper face of P. A simplicial complex is an example of a regular polytopal cell complex where each cell is a simplex, as is a Δ-complex. More generally, a regular polytopal cell complex is *piecewise Euclidean* (or PE) if its cells are each Euclidean polytopes and identifications are performed via isometries. Similarly, *piecewise spherical* (or PS) cell complexes are built out of spherical polytopes via isometric identification of faces. We add that a polytopal cell complex is *locally finite* if each cell is contained in the boundary of finitely many other cells.

Example 3.3.5 (Coxeter complexes)*.* If W is a spherical or Euclidean Coxeter group, then the associated Coxeter complex naturally has the structure of a piecewise spherical or piecewise Euclidean cell complex, respectively.

There is a strong relationship between these PE and PS complexes which may be seen via the following tool for certain cell complexes.

Definition 3.3.6 (Links)**.** Given a convex Euclidean polytope P in $\mathbb{E}^n$ and a vertex $\mathbf{v}$ of P, the *vertex link* of $\mathbf{v}$ in P (denoted $\textsc{Link}(\mathbf{v}, P)$) is the spherical polytope obtained by taking the intersection of P with an arbitrarily small $(n-1)$-sphere centered at $\mathbf{v}$, then rescaling so that the sphere has unit radius. In other words, it is the spherical polytope formed by the space of unit vectors $\mathbf{u}$ such that $\mathbf{v} + \epsilon\mathbf{u}$ is in P for some $\epsilon > 0$.

More generally, let f be a face of dimension k in P with barycenter $\mathbf{v}_f$. Then the *face link* of f in P (denoted $\textsc{Link}(f, P)$) is the space of unit vectors $\mathbf{u}$ such that $\mathbf{v}_f + \epsilon\mathbf{u}$ is in P for some $\epsilon > 0$ and, if $\mathbf{w}$ is a vector based at the barycenter $\mathbf{v}_f$ which points into f, then $\mathbf{u} \cdot \mathbf{w} = 0$. Then the face link of f in P is the intersection of a (rescaled) small $(k-1)$-sphere centered at $\mathbf{v}_f$ with P. For example, when P is a Euclidean tetrahedron and f is an edge of P, then the barycenter of f is simply the midpoint, and up to scaling there is a unique vector based at the barycenter which points into f - fix one and call it $\mathbf{w}$. The space of vectors pointing into P based at the barycenter of f and orthogonal to $\mathbf{w}$ (i.e. $\textsc{Link}(f, P)$) is then a spherical arc of length α, where α is the dihedral angle between the simplices which contain f as an edge.

If X is a piecewise Euclidean locally finite polytopal cell complex and $\mathbf{v}$ is a vertex in X, then $\mathbf{v}$ is incident to some finite number of polytopes. The *vertex link* of $\mathbf{v}$ in X (denoted $\textsc{Link}(\mathbf{v}, X)$) is the piecewise spherical complex formed by gluing together the vertex links of v in each of its incident polytopes in the natural way. In other words,

$\textsc{Link}(\mathbf{v}, X)$ is the ϵ-sphere in X based at $\mathbf{v}$, where $\epsilon > 0$ is smaller than each of the edge lengths incident to $\mathbf{v}$, then rescaled to $\epsilon = 1$. Hence, the vertex link inherits a cell structure and metric from X.

Similarly, the *face link* of a cell f in X (denoted $\textsc{Link}(f, X)$) is a piecewise spherical complex formed by the face links of f in each of the polytopes to which it belongs.

In general, the vertex links of a Euclidean polytope are spherical polytopes and the links in a PE complex are PS complexes. Moreover, this structure respects direct products in the sense of the following two definitions.

Definition 3.3.7 (Products of PE Complexes)**.** If X and Y are PE complexes, then the metric product $X \times Y$ is also piecewise Euclidean with an easily-defined metric: the distance between (x, y) and (a, b) in $X \times Y$ is given by the formula $\sqrt{d_X(x,a)^2 + d_Y(y,b)^2}$. Also, the cells in $X \times Y$ are polytopes of the form $\sigma \times \tau$, where σ and τ are polytopal cells in X and Y respectively.

When a PE complex decomposes as a direct product, its PS vertex links decompose as well, but not as direct products.

Definition 3.3.8 (Spherical Joins)**.** If K and L are PS complexes, then there are PE complexes X and Y with vertices $x \in X$ and $y \in Y$ such that K is the link of x in X and L is the link of y in Y. The *spherical join* $K \star L$ is given by the vertex link of (x, y) in the PE complex $X \times Y$. The cells in $K \star L$ are of the form $\sigma * \tau$, where σ and τ are faces of K and L, respectively. Notably, σ and τ are not required to be nonempty in this definition, thus providing copies of K and L within their spherical join $K \star L$ via faces of the form $\sigma \star \emptyset$ and $\emptyset \star \tau$.

3.4 Metric Spaces and Curvature

A main theme in this dissertation is to understand the connections between combinatorics of simplicial complexes and the curvature of their natural metrics. In this section we review a few definitions related to curvature and explore some of their applications in the topics discussed earlier in the chapter.

We begin by reviewing several definitions for geodesic metric spaces.

Definition 3.4.1 (Geodesics)**.** Let (X, d) denote a metric space X with metric function d. If $x, y \in X$, then a *geodesic* is an isometrically embedded interval of length $d(x, y)$ with endpoints at x and y. Then (X, d) is a *geodesic metric space* if every pair of points in X forms the endpoints of a geodesic. If (X, d) is a geodesic metric space with $x, y, z \in X$, then a *geodesic triangle* Δ on these points is a choice of geodesics $\mathrm{geod(x,y)}$, $\mathrm{geod(y,z)}$, $\mathrm{geod(z,x)}$ in X; refer to these as the *sides* of Δ and to x, y, z as the *vertices*.

As defined in Section 3.3, PE and PS complexes may be obtained by gluing together metric Euclidean or spherical polytopes, but in general these complexes do not have well-defined curvature properties. The following theorem due to Martin Bridson establishes a useful condition for when the local metrics on the cells combine to give a well-defined global metric which yields a geodesic metric space.

Theorem 3.4.2 ([BH99]). *If X is a* PE *complex or a* PS *complex whose cells have finitely many isometry types, then X is a complete geodesic metric space.*

Definition 3.4.3 (Geodesic Loops). A *geodesic interval* in a metric space is an isometrically embedded metric interval and a *geodesic loop* is an isometrically embedded metric circle. The weaker notion of *local geodesic interval* and *local geodesic loop* requires only that the interval and circle be local isometric embeddings, respectively. Finally, we say that a geodesic loop is *short* if it has length less than 2π.

We are interested not only in studying metric spaces, but also in the types of group actions they admit.

Definition 3.4.4 (Group actions). Let Γ be a group which acts on a metric space (X, d). Then Γ acts

- *properly discontinuously* if for each compact $K \subset X$, the set $\{g \in \Gamma \mid gK \cap K \neq \emptyset\}$ is finite;
- *cocompactly* if the quotient X/Γ is compact;
- *by isometries* if for all $g \in \Gamma$ and $x, y \in X$, we have $d(x, y) = d(gx, gy)$.

If the action of Γ on (X, d) satisfies all three of the above properties, then we say that Γ acts *geometrically* on (X, d).

Studying geometric group actions on metric spaces with interesting features is the core motivation for geometric group theorists. Among our most useful tools is the notion of nonpositive curvature, explored by Alexandrov in the 1950s and included in Gromov's definition of CAT(κ) spaces in 1987. In our applications, we need only to define the cases when $\kappa \in \{0, 1\}$. For a full treatment of these spaces, see the classic text by Bridson and Haefliger [BH99].

Definition 3.4.5 (Euclidean comparison triangles)**.** Let X be a geodesic metric space and let Δ be a geodesic triangle on vertices $x, y, z \in X$. Then there is a geodesic triangle Δ' in $\mathbb{E}^2$ on vertices a, b, c with the property that $d(x, y) = d(a, b)$, $d(y, z) = d(b, c)$, and $d(z, x) = d(c, a)$ and we say that Δ' is a *Euclidean comparison triangle* for Δ and observe that it is unique up to an isometry of $\mathbb{E}^2$.

There is an analogous definition for spherical comparison triangles, but one which requires an additional assumption.

Definition 3.4.6 (Spherical comparison triangles)**.** Let X be a geodesic metric space and let Δ be a geodesic triangle on vertices $x, y, z \in X$. If the side lengths of Δ add to less than 2π, then there is a geodesic triangle Δ' in the unit 2-sphere $\mathbb{S}^2$ on vertices a, b, c with the property that $d(x, y) = d(a, b)$, $d(y, z) = d(b, c)$, and $d(z, x) = d(c, a)$ and we say that Δ' is a *spherical comparison triangle* for Δ and observe that it is unique up to an isometry of $\mathbb{S}^2$.

Curvature of a geodesic metric space can be described by imposing restrictions on the "thickness" of geodesic triangles relative to their comparison triangles, when they exist. The two cases in which we are interested focus on metric spaces whose triangles which are "no thicker than" Euclidean or spherical triangles, leading to the definitions of CAT(0) and CAT(1) spaces respectively.

Definition 3.4.7 (CAT(0) spaces)**.** Let X be a geodesic metric space and fix a geodesic triangle Δ on vertices $x, y, z \in X$ with edges $\mathrm{geod(x,y)}$, $\mathrm{geod(y,z)}$, $\mathrm{geod(z,x)}$ in X. Let Δ' be a Euclidean comparison triangle for Δ and notice that, without loss of generality, if $p \in \mathrm{geod(x,y)}$ is a point on a side of Δ, there is a unique point $p' \in \mathrm{geod(a',b')}$ on the corresponding side of Δ' such that $d_X(x,p) = d_{\mathbb{E}^2}(a',p')$ and $d_X(p,y) = d_{\mathbb{E}^2}(p',b')$. If, for every pair of points p and q in the sides of Δ with corresponding points p' and q' in the sides of Δ', the inequality $d_X(p,q) \leq d_{\mathbb{E}^2}(p',q')$ holds, then Δ *satisfies the* CAT(0) *inequality.* We say X is a CAT(0) *space* if each of its geodesic triangles satisfies the CAT(0) inequality.

Definition 3.4.8 (CAT(1) spaces)**.** Let X be a geodesic metric space and fix a geodesic triangle Δ as above, but with the restriction that the edges $\mathrm{geod(x,y)}$, $\mathrm{geod(y,z)}$, and $\mathrm{geod(z,x)}$ have lengths which add to less than 2π. Then there is a spherical comparison triangle Δ' for Δ. If, for every pair of points p and q in the sides of Δ with corresponding points p' and q' in the sides of Δ', the inequality $d_X(p,q) \leq d_{\mathbb{S}^2}(p',q')$ holds, then Δ *satisfies the* CAT(1) *inequality.* We say X is a CAT(1) *space* if each of its geodesic triangles satisfies the CAT(1) inequality.

We can now record several useful facts on CAT(0) spaces. To begin, the property of being CAT(0) is preserved under direct products.

Lemma 3.4.9 ([BH99]). *If X and Y are* CAT(0) *spaces, then the direct product $X \times Y$ is also* CAT(0)*.*

As mentioned in [BH99], every CAT(0) space is contractible. For metric spaces with more interesting fundamental groups, there is the following local condition.

Definition 3.4.10 (Nonpositively curved spaces). If X is a geodesic metric space such that each point in X has a neighborhood which is a CAT(0) space, then X is *nonpositively curved.*

While nonpositive curvature is not the same as being CAT(0), the two coincide for certain spaces in their universal cover. The following theorem was given by Gromov in 1987, extending an older result for Riemannian manifolds with nonpositive sectional curvature - see [Bal95] for a proof.

Theorem 3.4.11 (Cartan-Hadamard). *If X is a complete, connected, nonpositively curved space, then its universal cover is* CAT(0)*.*

While we are primarily interested in spaces which are CAT(0), this type of curvature can be described via the CAT(1) conditions in the case of certain PE complexes. The following is the spherical analogue for nonpositive curvature.

Definition 3.4.12 (Locally CAT(1)). If X is a PS complex and every face link in X has no short geodesic loops, then X is *locally* CAT(1).

A proof of the following proposition may be found in [BH99].

Proposition 3.4.13 (Link Condition)**.** *If X is a* PE *complex whose cells have finitely many isometry types and the link of every face in X has no short geodesic loops, then X is nonpositively curved. Additionally, if Y is a locally* CAT(1) PS *complex with no short geodesic loops, then Y is* CAT(1)*. Furthermore, a contractible* PE *complex is* CAT(0) *if and only if its vertex links are* CAT(1)*.*

Groups with geometric actions on CAT(0) satisfy many pleasing properties and are a central motivator for this dissertation.

Definition 3.4.14 (CAT(0) group)**.** Let Γ be a group. If there is a complete CAT(0) space on which Γ acts geometrically, then Γ is a CAT(0) *group*.

Interest in CAT(0) groups stems primarily from the host of interesting properties which follow from the existence of a geometric action on a space with nonpositive curvature. Below is a handful of consequences for CAT(0) groups; a more comprehensive list may be found in [BH99].

Theorem 3.4.15 ([BH99])**.** *Let Γ be a* CAT(0) *group. Then:*

1. *Γ has solvable word and conjugacy problems.*

2. *Γ is finitely presented.*

3. *Γ has only finitely many conjugacy classes of finite subgroups.*

4. *Every solvable subgroup of Γ is virtually abelian.*

5. *Every abelian subgroup of Γ is finitely generated.*

Since the introduction of CAT(0) geometry, several classes of groups have been proven to be CAT(0). These include Coxeter groups [Mou88], right-angled Artin groups [CD95], and braid groups of graphs [Abr00]. Interestingly, the braid group has eluded such a theorem except in small rank. While $\textsc{Braid}_n$ is conjectured to be CAT(0) for all n [BM10], this has only been shown for $n \in \{3, 4, 5\}$ [BM10] and $n = 6$ [HKS16]. Each of these proofs make use of a metric on the *dual braid complex* defined by Brady and McCammond which forms the premise for this dissertation. We explore the properties of this space in Chapter 6.

The dual braid complex is a PE complex whose vertex links are PS complexes arising from a partially ordered set known as the *noncrossing partition lattice*. Much of the focus on proving that the dual braid complex is CAT(0) has revolved around using the combinatorics of this poset to determine the curvature of its associated *link complex*, defined in Section 3.6. See Chapter 5 for information on noncrossing partitions.

3.5 Orthoschemes and Columns

In this section we identify an interesting simplicial complex and explore some of its properties. See [BM10] and [DMW] for more references.

Consider the region in $\mathbb{R}^n$ which consists of all points $\mathbf{x} = \{x_1, \ldots, x_n\}$ with the property that

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq x_1 - 1.$$

Notice that this space has already appeared in Section 2.2, obtained via the hyperplanes of the form $x_i - x_{i+1} = 0$, together with the hyperplane determined by $x_1 - x_n = 1$. Notice also that this region is invariant under translation along the vector $\mathbf{1} = (1, \ldots, 1)$ and, as mentioned earlier, intersects the hyperplane $\mathbf{1}^\perp$ (defined by the equation $x_1 + \cdots + x_n = 0$) in the $\widetilde{A}_{n-1}$ Coxeter shape. In particular, this region splits as the direct product of $\widetilde{A}_{n-1}$ and $\langle \mathbf{1} \rangle = \mathbb{R}$.

Example 3.5.1 (Column in $\mathbb{R}^3$). The region $\mathcal{C}$ in $\mathbb{R}^3$ defined by the inequalities

$$x_1 \geq x_2 \geq x_3 \geq x_1 - 1$$

is bounded by the planes determined by $x_1 - x_2 = 0$, $x_2 - x_3 = 0$, and $x_1 - x_3 = 1$, and it is invariant under translation in the $(1, 1, 1)$ direction. The intersection of $\mathcal{C}$ with $(1, 1, 1)^\perp$, the plane determined by the equation $x_1 + x_2 + x_3 = 0$, is an equilateral triangle, also known as the Coxeter shape of type $\widetilde{A}_2$. The region $\mathcal{C}$ then decomposes as the metric direct product of $\mathbb{R}$ and the $\widetilde{A}_2$ Coxeter shape. We refer to $\mathcal{C}$ as a *standard* 3*-dimensional column* - see Figure 3.4.

The column described above can be endowed with a natural simplicial structure. Notice that the integer lattice points $(x_1, x_2, x_3) \in \mathbb{Z}^3$ which satisfy the inequalities $x_1 \geq x_2 \geq x_3 \geq x_1 - 1$ form a bi-infinite sequence as follows:

$$\ldots, (0, 0, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1), (2, 1, 1), \ldots$$

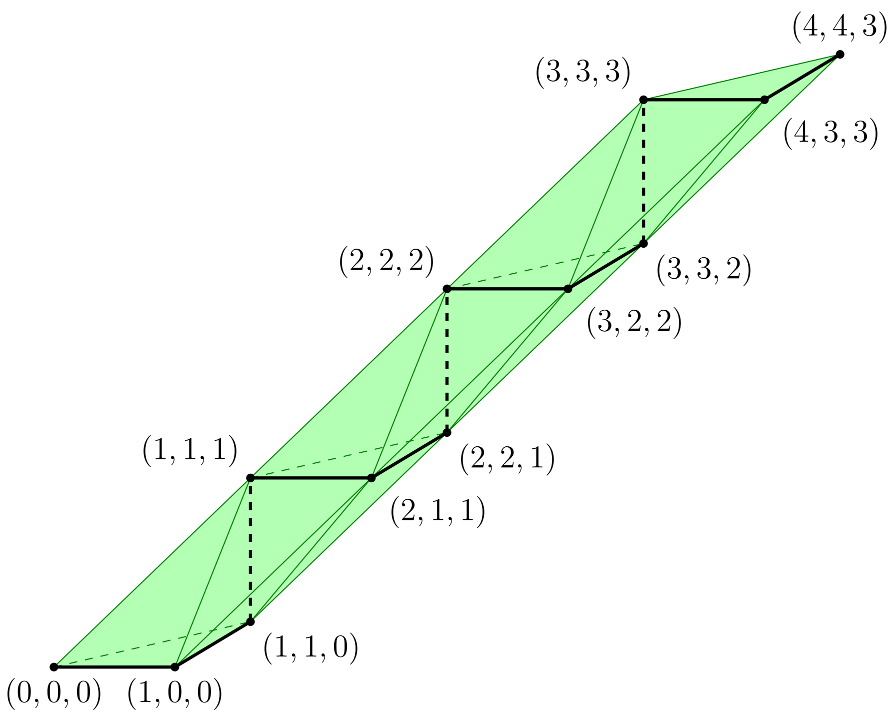

Figure 3.4: A column in $\mathbb{R}^3$

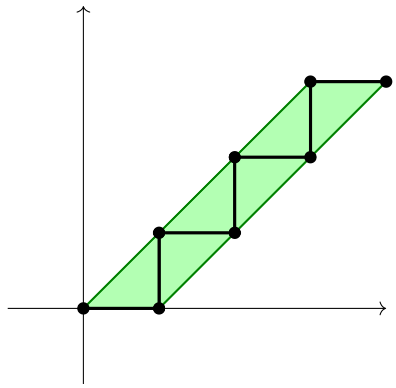

Figure 3.5: A portion of the vertices $(x_1, x_2) \in \mathbb{R}^2$ which satisfy the inequalities $x_1 \geq x_2 \geq x_1 - 1$, forming a region with a natural simplicial structure

The convex hull of any 4 consecutive vertices is an 3-simplex, and the set of all such simplices forms a simplicial structure for the 3-dimensional column. Notice, however, that the direct product structure mentioned above is not visible in this cell structure.

A special type of simplex arises from this construction. All 3-simplices obtained in the simplicial structure above are isometric, and this common isometry type is called an *orthoscheme*. These metric simplices were first described by Coxeter before being re-introduced by Brady and McCammond [BM10] and form an important building block in our study.

Definition 3.5.2 (Orthoschemes)**.** Let $p_0, \ldots, p_d$ be distinct vertices in $\mathbb{R}^n$ and define the vector $\mathbf{v}_i = p_i - p_{i-1}$ for each $i \in [d]$. Then the convex hull of $p_0, \ldots, p_d$ is a *standard d-dimensional orthoscheme* (or just *d-orthoscheme*) if $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ is an orthonormal set in $\mathbb{R}^n$.

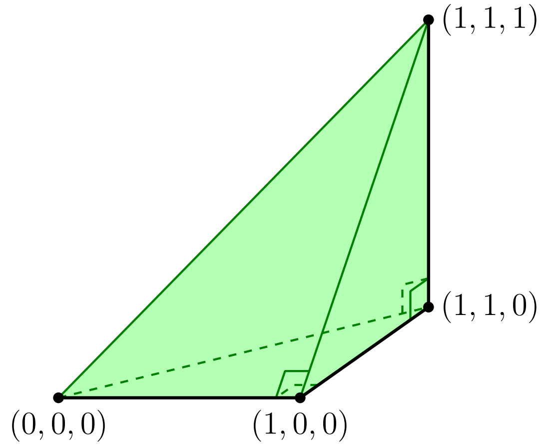

Figure 3.6: A 3-dimensional orthoscheme

Notice that the unit n-cube $[0,1]^n$ in $\mathbb{R}^n$ decomposes into $n!$ standard orthoschemes, each determined by a choice of path in the 1-skeleton from $(0,\ldots,0)$ to $(1,\ldots,1)$. Additionally, Examples 3.2.5 and 3.2.6 produce order complexes which are built out of orthoschemes. It is not too hard to see that a similar metric can be added in many situations.

Definition 3.5.3 (Orthoscheme Complex)**.** Let L be a graded locally finite locally ordered set. Then each ordered simplex in the order complex $\Delta(L)$ can be given a metric by requiring that it is an ordered Euclidean simplex with the length of the edge from $\mathbf{v}_1$ to $\mathbf{v}_2$ equal to $\sqrt{\mathrm{rk}[\mathbf{v}_1,\mathbf{v}_2]}$. Since Euclidean simplices are determined by their edge lengths, the result is a metric simplicial complex built out of orthoschemes which we refer to as $\textsc{Cplx}(L)$, the *orthoscheme complex* for L.

Among the key benefits of the orthoscheme metric is its interaction with products of ordered simplicial complexes.

Proposition 3.5.4 (Orthoscheme Products). *Let Δ_1 and Δ_2 be orthoscheme complexes. Then the ordered simplicial product $\Delta_1 \boxslash \Delta_2$ is an orthoscheme complex which is isometric to $\Delta_1 \times \Delta_2$.*

Proof. The projection maps $p_1 : \Delta_1 \boxslash \Delta_2 \to \Delta_1$ and $p_2 : \Delta_1 \boxslash \Delta_2 \to \Delta_2$ provide a map $\Delta_1 \boxslash \Delta_2 \to \Delta_1 \times \Delta_2$ given by (p_1, p_2), and by Proposition 3.2.10, this is a homeomorphism. All that remains to show is that the metric is preserved, and it suffices to check this on the vertices of $\Delta_1 \boxslash \Delta_2$.

Notice that the covering relations in the local order associated to $\Delta_1 \boxslash \Delta_2$ are those of the form $(u, v) < (u', v)$ and $(u, v) < (u, v')$, where $u < u'$ and $v < v'$ are covering relations in the local orders for Δ_1 and Δ_2, respectively. Since all the local orders in question are graded, this tells us that whenever $(u, v) < (u', v')$ we know that the rank $\mathrm{rk}[(u, v), (u', v')]$ is equal to the sum $\mathrm{rk}[u, u'] + \mathrm{rk}[v, v']$. Thus, by Definition 3.5.3, edge lengths are preserved and therefore the map (p_1, p_2) is an isometry. $\square$

Notice that if we consider $\mathbb{R}$ as an infinite simplicial directed graph with vertex set $\mathbb{Z}$, then the n-fold orthoscheme product $\mathbb{R} \boxslash \cdots \boxslash \mathbb{R}$ yields the *orthoscheme tiling* of $\mathbb{R}^n$ described in Example 3.2.6.

Returning to our motivating example, the component of $\mathbb{R}^n$ described by the inequalities $x_1 \geq x_2 \geq \cdots \geq x_n \geq x_1 - 1$ can be given a cell structure where the maximal simplices are n-dimensional orthoschemes. Since this space is the direct product of a

Euclidean simplex and $\mathbb{R}$, Brady and McCammond refer to it as a *column* [BM10]. We can make this and related notions precise with the following definitions.

The example above can be made far more general, allowing for any dimension and choice of origin or region.

Definition 3.5.5 (Columns)**.** Consider the region $\mathcal{C}$ of $\mathbb{R}^k$ defined by the inequalities

$$x_{\pi(1)} - a_{\pi(1)} \geq x_{\pi(2)} - a_{\pi(2)} \geq \cdots \geq x_{\pi(k)} - a_{\pi(k)} \geq x_{\pi(1)} - a_{\pi(1)} - 1$$

where $\pi \in \textsc{Sym}_k$ and $(a_1, \ldots, a_k) = \mathbf{a} \in \mathbb{R}^k$. Metrically, this region contains the line through the point $\mathbf{a}$ spanned by $\mathbf{1} = (1, \ldots, 1)$ and is invariant under translation in the $\mathbf{1}$ direction. In particular, $\mathcal{C}$ splits as the direct product of $\langle \mathbf{1} \rangle = \mathbb{R}$ and a $\widetilde{A}_{k-1}$ Coxeter shape. Inspired by this structure, we refer to $\mathcal{C}$ as a *column* in $\mathbb{R}^k$. Furthermore, observe that the integer lattice points in this region appear only on the boundary and may be ordered in a bi-infinite sequence. Specifically, these points form a set $L = \{\mathbf{v}_l \mid l \in \mathbb{Z}\}$ with the property that the standard inner product $\langle \mathbf{v}_l, \mathbf{1} \rangle = l$. We may define a local order $\leq_L$ on L by declaring that $\mathbf{v}_i \leq_L \mathbf{v}_j$ if $j - i \leq k$. Then the orthoscheme complex $\textsc{Cplx}(L)$ is isometric to $\mathcal{C}$, endowing the column with a natural orthoscheme structure. Notice that this is a subcomplex of the orthoscheme tiling for $\mathbb{R}^k$ described previously.

We may further generalize the notion of column to include a version which has been scaled to be larger.

Definition 3.5.6 (Dilated Columns)**.** The column in Definition 3.5.5 is given by introducing the hyperplane defined by the equation $x_{\pi(1)} - x_{\pi(k)} = 1$ to a copy of the real braid arrangement and considering the resulting region. If, for some positive integer ℓ, we instead use the hyperplane given by $x_{\pi(1)} - x_{\pi(n)} = \ell$, we obtain a larger region $\mathcal{C}$ which we refer as a *dilated column*. For any $\pi \in \textsc{Sym}_n$ and $(a_1, \ldots, a_n) = \mathbf{a} \in \mathbb{R}^n$, we then define the associated dilated column to be the full subcomplex of the orthoscheme tiling of $\mathbb{R}^n$ which is determined by the following inequalities:

$$x_{\pi(1)} - a_{\pi(1)} \geq x_{\pi(2)} - a_{\pi(2)} \geq \cdots \geq x_{\pi(k)} - a_{\pi(k)} \geq x_{\pi(1)} - a_{\pi(1)} - \ell$$

These regions have a similar product decomposition into $\mathbb{R}$ and a Coxeter shape of type $\widetilde{A}_{k-1}$ which has been dilated by a factor of ℓ. In particular, one can use a standard volume calculation to show that the full region can be viewed as the union of ℓ^{k-1} columns, which together give $\mathcal{C}$ a tiling by orthoschemes.

Definition 3.5.7 ((k,n)-dilated column)**.** Let $n > k$ be positive integers and let $\mathcal{C}$ be the full subcomplex of the orthoscheme tiling of $\mathbb{R}^k$ where the vertex set consists of those vertices satisfying the strict inequalities

$$x_1 > x_2 > \cdots > x_k > x_1 - n.$$

Then $\mathcal{C}$ is referred to as a (k,n)-*dilated column* in $\mathbb{R}^k$. While these inequalities don't immediately resemble those of Definition 3.5.6, we may rewrite them as follows. By repeated use of the trivial observation that for integers a and b, we have $a > b$ if and only if $a \geq b + 1$, we may see that the inequalities above are equivalent on integer lattice points to the following:

$$x_1 + 1 \geq x_2 + 2 \geq \cdots \geq x_k + k \geq x_1 + 1 - (n - k).$$

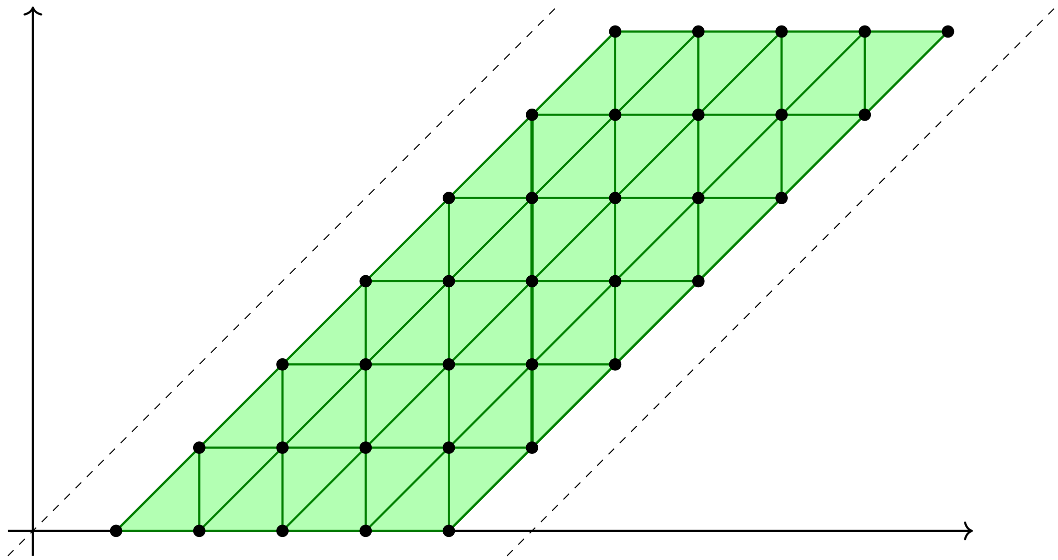

Figure 3.7: Part of the $(2,6)$-dilated column described in Example 3.5.8

We may then see that $\mathcal{C}$ is isometric to a dilated column in the sense of Definition 3.5.6 with $\ell = n - k$. Choose $\pi \in \mathrm{S{\small YM}}_k$ to be the identity and $\mathbf{a} \in \mathbb{R}^k$ by $a_i = -i$. Then the inequalities above match those in Definition 3.5.6 and thus each (k,n)-dilated column is appropriately named. As in Definition 3.5.6, $\mathcal{C}$ is the union of $(n-k)^{k-1}$ columns.

Example 3.5.8 ($(2,6)$-dilated column). The $(2,6)$-dilated column is a union of $(6-2)^{(2-1)}$ columns in $\mathbb{R}^2$ and determined by the integer lattice points which satisfy the inequalities $x_1 > x_2 > x_1 - 6$ (or equivalently $x_1 + 1 \geq x_2 + 2 \geq x_1 - 3$), depicted in Figure 3.7.

With the tools developed in Section 3.4, we have the following proposition, which is used in Chapter 7.

Proposition 3.5.9. *Every (k,n)-dilated column is* $\mathrm{CAT}(0)$.

Proof. As described in Definition 3.5.7, each dilated column is isometric to the direct product of $\mathbb{R}$ and a dilated Coxeter shape of type $\widetilde{A}_{k-1}$. The former is clearly CAT(0) and the latter is a closed convex subspace of the $\widetilde{A}_{k-1}$ Coxeter complex, which is a tiling of $\mathbb{R}^{k-1}$ by Coxeter shapes. Hence, both terms of the product are CAT(0) and by Lemma 3.4.9, the product is CAT(0). Hence, dilated columns are CAT(0). □

3.6 Poset Topology

Checking directly for nonpositive curvature in a geodesic metric space is difficult in general, but there are several useful tools when we restrict to piecewise Euclidean complexes. We are particularly interested in one which relates the curvature of a PE complex to the curvature of its vertex links.

Consider the case of a bounded poset with unique minimum and maximum elements $\hat{0}$ and $\hat{1}$ respectively. As mentioned in the preceding section, the order complex of this poset is contractible. Instead, consider the edge from $\hat{0}$ to $\hat{1}$ in the order complex $\Delta(P)$. If we can put a piecewise Euclidean metric on $\Delta(P)$, then the link of this edge contains the same combinatorial information as $\Delta(P)$ in the following sense. For a full treatment, see [Wac07].

Definition 3.6.1 (Poset Links)**.** Suppose P is a bounded graded finite poset with unique minimum and maximum elements $\hat{0}$ and $\hat{1}$ respectively. Then the maximal chains in P each begin at $\hat{0}$ and end at $\hat{1}$, so the maximal simplices in the order complex $\Delta(P)$ each contain the edge between the minimum and maximum elements. Define the *poset link*

$\textsc{Link}(P)$ to be the order complex of the partially ordered set $\bar{P} = P - \{\hat{0}, \hat{1}\}$. There is an obvious bijection between the maximal chains of P and $\bar{P}$ obtained by removing $\hat{0}$ and $\hat{1}$, but while $\Delta(P)$ is contractible, $\textsc{Link}(P) = \Delta(\bar{P})$ is generally not. Moreover, when we endow $\Delta(P)$ with the (piecewise Euclidean) orthoscheme metric, $\textsc{Link}(P)$ may be viewed as the link of the edge from $\hat{0}$ to $\hat{1}$ and thus inherits a natural piecewise spherical metric.

Consider the following instructive example.

Example 3.6.2 (Boolean lattice). Let $\textsc{Bool}_n$ be the *Boolean lattice of rank* n, i.e. the power set of $[n]$, ordered by inclusion. Since there are $n!$ maximal chains in this poset, the order complex consists of $n!$ n-simplices, all of which share the edge which connects the minimum and maximum elements. In particular, $\Delta(\textsc{Bool}_n)$ is isometric to the unit n-cube when given the orthoscheme metric described above.

The longest edge in this complex connects the vertices representing the minimum and maximum elements in $\textsc{Bool}_n$, and the link of this edge in the order complex is an $(n-2)$-sphere which has been subdivided into $n!$ spherical simplices. In fact, this edge link is isometric to the A_{n-1} Coxeter complex. Topologically, this complex is homeomorphic to the order complex of the Boolean lattice with its bounding elements removed.

We conclude this chapter with the following remark.

Remark 3.6.3. If P and Q are bounded graded finite posets, with orthoscheme complexes $\Delta(P)$ and $\Delta(Q)$, then we know by Proposition 3.2.10 that $\Delta(P \times Q)$ is isometric to

$\Delta(P) \times \Delta(Q)$. Together with Definition 3.3.8, we observe that the poset link for $\Delta(P \times Q)$ is isometric to the spherical join of the poset links for $\Delta(P)$ and $\Delta(Q)$.

4. Configuration Spaces

While we have described the braid group up to this point as an Artin group, there is another important perspective to consider. As described in Section 2.1, the braid group is the fundamental group of a configuration space, a special type of topological space which we describe in this chapter.

Configuration spaces are generally defined as arising from arbitrary topological spaces, but analogues exist for certain types of structures. For example, Aaron Abrams defined a variant of configuration spaces for graphs in which the resulting topological space is a cube complex. With the natural cubical metric, Abrams shows that these spaces are nonpositively curved [Abr00]. In this chapter we discuss the characterization of braids via configuration spaces and describe a new type of configuration space for ordered simplicial complexes which carries a natural orthoscheme metric.

4.1 Topological Configuration Spaces

We begin with the classical definition of configuration spaces and a few examples.

Definition 4.1.1 (Configuration Space)**.** Let X be a topological space and n a positive integer. Then the *(ordered) configuration space of n points in X* is the space of all n-tuples of distinct points in X, denoted $\textsc{Conf}_n(X)$. In other words,

$$\textsc{Conf}_n(X) = X^n - \textsc{Diag}_n(X)$$

where we define the *thick diagonal* to be

$$\mathrm{D{\scriptstyle IAG}}_n(X) = \{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j \text{ for some } 1 \leq i < j \leq n\}.$$

The action of $\mathrm{S{\scriptstyle YM}}_n$ on X^n restricts to a free action on $\mathrm{C{\scriptstyle ONF}}_n(X)$ and the quotient $\mathrm{C{\scriptstyle ONF}}_n(X)/\mathrm{S{\scriptstyle YM}}_n$ is the *unordered configuration space of* n *points in* X, denoted $\mathrm{UC{\scriptstyle ONF}}_n(X)$. Just as an element in $\mathrm{C{\scriptstyle ONF}}_n(X)$ is an n-tuple of distinct points, an element in $\mathrm{UC{\scriptstyle ONF}}_n(X)$ is a *set* of n distinct points in X. In other words, the difference between the two is whether the n points are ordered or not.

There are several configuration spaces which are already well-known to us.

Example 4.1.2 (2 points in S^1). The configuration space $\mathrm{C{\scriptstyle ONF}}_2(S^1)$ of two points in the circle is the torus $T^2 = S^1 \times S^1$ with a $(1,1)$-curve removed, which can be viewed as the interior of an annulus. The action by $\mathrm{S{\scriptstyle YM}}_2 = \mathbb{Z}_2$ swaps the meridian and longitude curves of T^2 and thus the quotient $\mathrm{UC{\scriptstyle ONF}}_2(S^1)$ is the interior of a Möbius band.

Example 4.1.3 (n points in S^1). As the generalization of the above example, consider the configuration space of n points in S^1. The thick diagonal consists of the points in the n-torus $T^n = (S^1)^n$ with non-distinct entries. By considering the cases where two coordinates are equal, the configuration space $\mathrm{C{\scriptstyle ONF}}_n(S^1)$ is the n-torus minus the union of $\binom{n}{2}$ copies of an $(n-1)$-torus.

Example 4.1.4 (n points in $\mathbb{D}^2$). As pointed out in Section 2.1, the configuration space $\mathrm{C{\scriptstyle ONF}}_n(\mathbb{D}^2)$ is homotopy equivalent to the complement of the complex braid arrangement and hence has fundamental group isomorphic to the pure braid group. Similarly, the unordered configuration space $\mathrm{UC}\textsc{onf}_n(\mathbb{D}^2)$ has the braid group as its fundamental group.

In each of the above examples, both X and $\textsc{Conf}_n(X)$ are manifolds (and hence $\mathrm{UC}\textsc{onf}_n(X)$ is as well), but when X carries a different structure (e.g. as a graph or a simplicial complex), there is an alternative approach which takes this information into account.

4.2 Basepoints and Strands

Our primary use of configuration spaces in this dissertation is to represent elements of the braid group. In this section, we establish our conventions for these representations, largely following [DMW].

Since each braid in $\textsc{Braid}_n$ is represented by a loop in $\mathrm{UC}\textsc{onf}_n(\mathbb{D}^2)$, different choices of basepoint in the latter lead to different natural elements in the former. More specifically, certain motions are more easily described if our n points begin at a well-chosen basepoint. As will become clear in later chapters, we make frequent use of a braid which cyclically rotates all strands - as such, it behooves us to arrange them in a circle.

Definition 4.2.1 (Vertices)**.** Let $\mathbb{D}^2$ be the unit disk in $\mathbb{C}$ and fix an integer $n > 1$. Denote by P_n the set

$$P_n = \{e^{2k\pi i/n} \mid k \in \mathbb{Z}\}$$

where we further define $p_k = e^{2k\pi i/n}$ and note that $p_k = p_{k+n}$. These n points lie on the boundary of $\mathbb{D}^2$ and the n-tuple $\vec{p}_n = (p_1, \ldots, p_n)$ gives a basepoint for the ordered configuration space $\textsc{Conf}_n(\mathbb{D}^2)$. Similarly, the set $P_n = \{p_1, \ldots, p_n\}$ defines a basepoint for the unordered counterpart, $\mathrm{U}\textsc{Conf}_n(\mathbb{D}^2)$. For the remainder of this dissertation, we assume these basepoints when discussing either of these configuration spaces.

We note that even though our basepoint is chosen to be on the boundary of the disk and the points are only allowed to move in the closed disk $\mathbb{D}^2$, we will often draw a larger disk in the figures for the sake of visual clarity.

Definition 4.2.2 (Polygons)**.** Let $n > 1$ be an integer. Then the vertex set P_n given in Definition 4.2.1 consists of n points on the unit circle in $\mathbb{C}$, and the convex hull of these vertices is a regular n-gon centered at the origin. We refer to this polygon as D_n and observe that D_n is homeomorphic to the disk $\mathbb{D}^2$. In this dissertation, we will typically use D_n when discussing the configuration space of the disk; this will be of some importance in Chapter 7.

When considering a configuration space of points in $\mathbb{C}$, it suffices to consider the configurations which remain in the closed disk.

Proposition 4.2.3 (Deformation retraction)**.** *The configuration space* $\textsc{Conf}_n(\mathbb{D}^2)$ *of* n *points in the closed disk* $\mathbb{D}^2$ *is a deformation retract of* $\textsc{Conf}_n(\mathbb{C})$*.*

Proof. For each $\mathbf{z} = (z_1, \ldots, z_n)$ in $\textsc{Conf}_n(\mathbb{C})$, define $m(\mathbf{z}) = \max\{|z_1|, \ldots, |z_n|, 1\}$ and observe that m is a continuous function on this configuration space. Then $\frac{\mathbf{z}}{m(\mathbf{z})}$ lies in the unit disk $\mathbb{D}^2$ and we may define the explicit deformation retraction $f_t : \mathbb{C} \to \mathbb{D}^2$ as follows:

$$\begin{aligned} f_t(\mathbf{z}) &= (1-t)\mathbf{z} + t\frac{\mathbf{z}}{m(\mathbf{z})} \\ &= \frac{m(\mathbf{z}) + (1-m(\mathbf{z}))t}{m(\mathbf{z})}\mathbf{z} \end{aligned}$$

Moreover, since $m(\mathbf{z})$ is unaffected by the order of entries in $\mathbf{z}$, this map descends to a deformation retraction of $\mathrm{UC{\small ONF}}_n(\mathbb{C})$ to $\mathrm{UC{\small ONF}}_n(\mathbb{D}^2)$. $\square$

We may further restrict our attention from $\mathbb{D}^2$ to D_n via the following straightforward proposition.

Proposition 4.2.4. *Each homeomorphism $X \to X'$ induces a homeomorphism*

$$\mathrm{UC{\small ONF}}_n(X) \to \mathrm{UC{\small ONF}}_n(X')$$

and if this homeomorphism sends the basepoint $P \in \mathrm{UC{\small ONF}}_n(X)$ to $P' \in X'$, we obtain an isomorphism of fundamental groups

$$\pi_1(\mathrm{UC{\small ONF}}_n(X), P) \cong \pi_1(\mathrm{UC{\small ONF}}_n(X'), P').$$

Hence, the braid group $\mathrm{B{\small RAID}}_n$ may be identified with $\pi_1(\mathrm{UC{\small ONF}}_n(D_n), P_n)$.

Since $\mathrm{B{\small RAID}}_n$ is the fundamental group of the unordered configuration space of n points in D_n, each braid can be represented by a loop in $\mathrm{UC{\small ONF}}_n(D_n)$ which begins and ends at our chosen basepoint.

Remark 4.2.5 (Braid groups). By Propositions 4.2.3 and 4.2.4, $\mathrm{B{\small RAID}}_n$ is the fundamental group $\pi_1(\mathrm{UC{\small ONF}}_n(D_n))$. We specifically consider $\mathrm{B{\small RAID}}_n$ to be fundamental group of

$\mathrm{UC}\textsc{onf}_n(D_n)$ with basepoint P_n, and denote this $\pi_1(\mathrm{UC}\textsc{onf}_n(D_n), P_n)$. Throughout the remainder of this work, we will assume this picture when discussing $\mathrm{B}\textsc{raid}_n$.

Each loop in the unordered configuration space based at P_n may be lifted to a path in $\mathrm{C}\textsc{onf}_n(D_n)$ from the n-tuple $(p_1, \ldots, p_n)$ to $(q_1, \ldots, q_n)$, where $(q_1, \ldots, q_n)$ is a rearrangement of $(p_1, \ldots, p_n)$. We can record the data involved in this picture with the following two definitions.

Definition 4.2.6 (Strands)**.** Let $\beta \in \mathrm{B}\textsc{raid}_n$ be a braid with representative f. A *strand of* f is a path in D_n that follows what happens to one of the vertices in P_n. There are two natural ways to name strands: by where they start and by where they end. Let $\widetilde{f}^{\vec{p}_n}$ be the unique lift of f through the covering map $\mathrm{C}\textsc{onf}_n(D_n) \to \mathrm{UC}\textsc{onf}_n(D_n)$ with $\widetilde{f}^{\vec{p}_n}(0) = \vec{p}_n$. Then the *strand that starts at* p_i is the path $f^i \colon [0,1] \to D_n$ defined by the composition $f^i = \textsc{proj}_i \circ \widetilde{f}^{\vec{p}_n}$, where $\textsc{proj}_i$ is projection onto the i-th coordinate. Similarly, the *strand that ends at* p_j is the path $f_j \colon [0,1] \to D_n$ defined by the composition $f_j = \textsc{proj}_j \circ \widetilde{f}_{\vec{p}_n}$ where $\widetilde{f}_{\vec{p}_n}$ is the unique lift of the path f through the covering map $\mathrm{C}\textsc{onf}_n(D_n) \to \mathrm{UC}\textsc{onf}_n(D_n)$ so that ends at $\vec{p}_n$, i.e. $\widetilde{f}_{\vec{p}_n}(1) = \vec{p}_n$. When the strand of f that starts at p_i ends at p_j the path f^i is the same as the path f_j. We write f^i, f_j or f^i_j for this path and we call it the $(i,\cdot)$-*strand*, the $(\cdot, j)$-*strand* or the (i,j)-*strand of* f depending on the information specified.

Definition 4.2.7 (Drawing braids)**.** Let $\beta \in \mathrm{B}\textsc{raid}_n$ with representative f. Each strand in f is a path $[0,1] \to D_n$ which may be graphed in the product $[0,1] \times D_n$. Since f is a loop in $\mathrm{UC}\textsc{onf}_n(D_n)$, its strands have graphs which are pairwise disjoint embeddings of an interval in the polygonal prism $[0,1] \times D_n$. The *drawing* of f is given by these n embeddings. We draw this prism in $\mathbb{R}^3$ by embedding D_n in the xy-plane and orienting the $[0,1]$ component in the z direction so that $t=0$ is the "top" of our drawing and $t=1$ is the "bottom".

Remark 4.2.8 (Multiplication). Let β_1 and β_2 be braids in $\textsc{Braid}_n$ with representatives f_1 and f_2. The product $\alpha_1 \cdot \alpha_2$ is defined to be $[f_1 \cdot f_2]$ where $f_1 \cdot f_2$ is the concatenation of f_1 and f_2. The drawing for $f_1 \cdot f_2$ is obtained by stacking the drawing for f_1 on top of the drawing for f_2 and rescaling. See Figure 4.1 for an example.

Definition 4.2.9 (Braids and permutations)**.** Let $\beta \in \textsc{Braid}_n$ with representative f. The *permutation* of β is the bijection $\textsc{Perm}(\beta)\colon [n] \to [n]$ that sends $j \in [n]$ to the index of the vertex at the start of the $(\cdot, j)$-strand of β, i.e. to i if $f_j(0) = p_i \in P_n$. Note that the function $\textsc{Perm}(\beta)$ only depends on the braid β and not on the representative f. The direction of the bijection $\textsc{Perm}(\beta)$ is defined so that it is compatible with function composition, i.e. so that $\textsc{Perm}(\beta_1 \cdot \beta_2) = \textsc{Perm}(\beta_1) \circ \textsc{Perm}(\beta_2)$.

The convex hull of the vertices in P_n is a convex n-gon; it will be useful to articulate the same for subsets of P_n.

Definition 4.2.10 (Subsets of P_n)**.** Let $A \subseteq [n]$. Define P_A to be the subset of P_n given by $\{p_i \mid i \in A\}$ and let $\textsc{Conv}(P_A)$ be the convex hull of the vertices in P_A. Notice that if $|A| = k$, then $\textsc{Conv}(P_A)$ is a convex k-gon.

Each subset of P_n corresponds to a subgroup of $\textsc{Braid}_n$.

Definition 4.2.11 (Fixed strands)**.** Let $\beta \in \textsc{Braid}_n$. For each $b \in [n]$, we say that β *fixes* the vertex p_b if there is a representative f of β such that the $(b, \cdot)$ strand is a constant path, i.e. $f^b(t) = p_b$ for all $t \in [0, 1]$. Similarly, for each $B \subseteq [n]$, we say that β *fixes* the vertex set P_B if there is a single representative f of β which fixes each $p_b \in P_B$. Define

$$\textsc{Fix}_n(B) = \{\beta \in \textsc{Braid}_n \mid \beta \text{ fixes } P_B\}$$

and observe that, since concatenation and inversion of the required representatives preserves their fixed strands, $\textsc{Fix}_n(B)$ is a subgroup of $\textsc{Braid}_n$.

Each subset of P_n determines a *subdisk* of D_n.

Definition 4.2.12 (Subdisks of D_n)**.** For all distinct $i, j \in [n]$ let the *edge* e_{ij} be the straight line segment connecting p_i and p_j. For $k > 2$, let D_A be $\textsc{Conv}(P_A)$, the convex hull of the points in P_A and note that D_A is a k-gon homeomorphic to $\mathbb{D}^2$. We call this the *standard subdisk for* $A \subseteq [n]$. In this notation, our original disk D_n is $D_{[n]}$. For $k = 2$ and $A = \{i, j\}$, we define D_A so that it is also a topological disk. Concretely, we take two copies of the path along the edge $e = e_{ij}$ from p_i to p_j and then bend one or both of these copies so that they become injective paths from p_i to p_j with disjoint interiors which together bound a bigon inside of D_n. Moreover, when the edge e lies in the boundary of D_n we require that one of the two paths does not move so that e itself is part of the boundary of the bigon. For $k = 1$, we define D_A to be the single point $p_i \in P_A$, but note that this subspace is not a subdisk. The bending of the edges to form the bigons are chosen to be slight enough so that for all A and $B \subseteq [n]$ the standard disks D_A and D_B intersect if and only if the convex hulls $\textsc{Conv}(P_A)$ and $\textsc{Conv}(P_B)$ intersect.

With the above conventions, we may now define some natural motions of our n points in the disk.

Definition 4.2.13 (Rotations)**.** Let $A \subseteq [n]$. We define the *rotation braid* δ_A by taking the vertices in P_A and rotating them counterclockwise around the boundary of the subdisk D_A so that each vertex travels along a single edge until it reaches the next vertex. If $|A| = 1$, then we let δ_A denote the identity braid. If $|A| = 2$, then D_A is a bigon as defined above and the braid obtained by counterclockwise rotation about its boundary is called a *positive half-twist*. When $A = [n]$, we write δ_n to represent δ_A. In general, observe that if $|A| = k$, then $\textsc{Perm}(A)$ is the k-cycle permutation obtained by ordering the elements of A in increasing order.

We may immediately observe that rotations δ_C and δ_D commute if $\textsc{Conv}(C)$ and $\textsc{Conv}(D)$ are disjoint. It is not difficult to show that this sufficient condition is also necessary - see Section 5.5 for further investigation of these elements.

Remark 4.2.14. The set of all rotation braids $\{\delta_A \mid A \subseteq [n]\}$ forms a generating set for $\textsc{Braid}_n$ since, in particular, it contains all of the positive half-twists.

Of particular interest are the ways that rotation braids may be factored into other rotation braids.

Example 4.2.15 (Rotations). Consider the subsets $C = \{1, 5, 6\}$ and $D = \{2, 3, 4, 5\}$ of $[6]$. Then $\delta_C\delta_D$ is also a rotation braid, namely δ_6 - see Figure 4.1. This property is

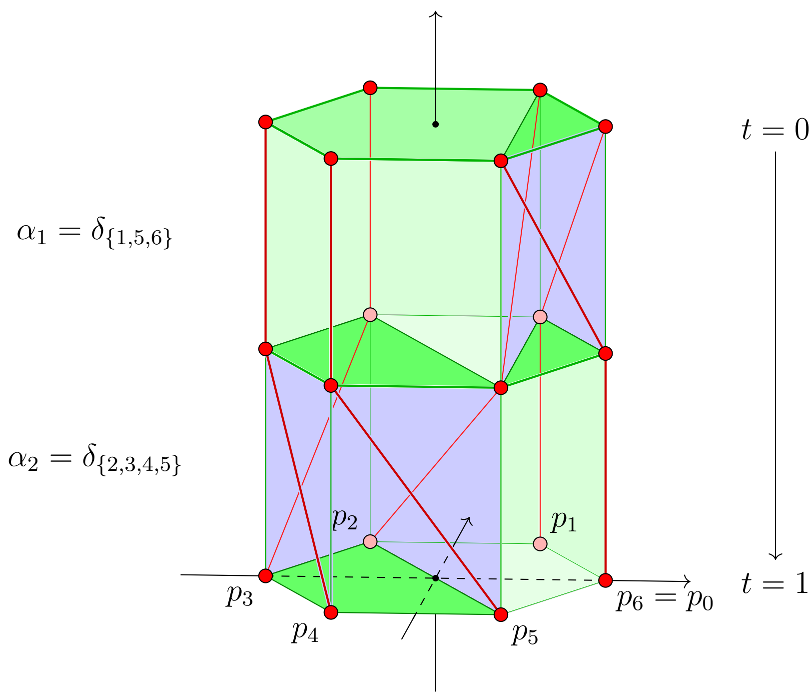

Figure 4.1: A drawing of $\delta_{\{1,5,6\}}\delta_{\{2,3,4,5\}}$, as described in Example 4.2.15

related to the fact that C and D are *properly ordered*, a notion to which we return in Definition 5.3.2.

Notice that with our particular choice of basepoint, the boundary cycle of the n-gon D_n can be viewed as a directed graph with n vertices and n edges, oriented counter-clockwise. In later chapters, we characterize braids with strands which remain in the boundary, necessitating an understanding of the configuration spaces for graphs.

4.3 Graphs and Orthoschemes

When X is a cell complex, then the k-fold product X^k also has a cell structure, but the thick diagonal $\text{DIAG}_n(X)$ is not generally a subcomplex. Hence we need a different definition for $\text{CONF}_k(X)$ if we are to make use of the structure for X. We are primarily interested in the cases when X is either a graph or an orthoscheme complex, so we discuss the techniques in these cases.

Definition 4.3.1 (Products)**.** If Γ is a directed graph, then there are two natural cell structures we may give to the k-fold product Γ^k. If we make each edge of Γ undirected, then Γ^k can be considered as a cube complex. If we preserve the directed edges, then Γ is a Δ-complex and the product space Γ^k can also be given a Δ-complex structure by Proposition 3.2.10. To distinguish between these homeomorphic structures, we use the notations $\text{PROD}_k(\Gamma, \square)$ and $\text{PROD}_k(\Gamma, \boxslash)$ respectively. Notice that if Γ is a metric directed graph with unit length edges, then $\text{PROD}_k(\Gamma, \square)$ is comprised of unit cubes and $\text{PROD}_k(\Gamma, \boxslash)$ is an orthoscheme complex.

Notice that the Δ-complex $\textsc{Prod}(\Delta, \boxslash)$ can be defined when Δ is an arbitrary Δ-complex.

Example 4.3.2. Let Γ_6 be the directed graph obtained as the boundary of a regular hexagon with unit length edges, oriented counter-clockwise. The product $\textsc{Prod}_2(\Gamma_6, \square)$ is a cell complex consisting of 36 squares which is homeomorphic to the 2-dimensional torus. Similarly, $\textsc{Prod}_2(\Gamma_6, \boxslash)$ is a simplicial complex consisting of 72 isoceles right triangles, obtained by subdividing each square in $\textsc{Prod}_2(\Gamma_6, \square)$ into two triangles.

For each type of product above, we may define the corresponding configuration space to be the largest subcomplex which avoids the thick diagonal. The two perspectives yield the following two definitions.

Definition 4.3.3 (Cubical configuration spaces)**.** Let Γ be a directed graph and k a positive integer. Then denote by $\textsc{Conf}_k(\Gamma, \square)$ the *cubical configuration space of k points in* Γ, which we define to be the largest subcomplex of $\textsc{Prod}_k(\Gamma, \square)$ which does not intersect the thick diagonal. The corresponding *unordered configuration space* $\text{U}\textsc{Conf}_k(\Gamma, \square)$ is given via the natural quotient by $\textsc{Sym}_n$. Both $\text{U}\textsc{Conf}_k(\Gamma, \square)$ and $\textsc{Conf}_k(\Gamma, \square)$ may be given a metric by making each cube in its cell structure a unit cube.

Definition 4.3.4 (Orthoscheme configuration spaces)**.** Let Γ be a directed graph (viewed as a 1-dimensional Δ-complex) and let k be a positive integer. We define the *orthoscheme configuration space of k points in* Γ to be the largest subcomplex of the Δ-complex $\textsc{Prod}_k(\Gamma, \boxslash)$ which does not intersect the thick diagonal. Denote this space by $\textsc{Conf}_k(\Gamma, \boxslash)$. The corresponding *unordered configuration space* $\text{U}\textsc{Conf}_k(\Gamma, \boxslash)$ is given via the natural quotient by $\textsc{Sym}_n$. Observe that even if Γ is a 1-dimensional ordered simplicial complex. then $\textsc{Conf}_k(\Gamma, \boxslash)$ is also an ordered simplicial complex, but $\textsc{UConf}_k(\Gamma, \boxslash)$ is only a Δ-complex.

Following the introduction of cubical configuration spaces in his 2000 dissertation, Aaron Abrams proved the following remarkable result on the curvature of these spaces [Abr00].

Theorem 4.3.5 ([Abr00])**.** *Cubical configuration spaces of graphs are nonpositively curved.*

The theorem above immediately prompts two questions.

Question 4.3.6. Are orthoscheme configuration spaces nonpositively curved?

Question 4.3.7. For which graphs Γ and positive integers k do we have a homotopy equivalence

$$\textsc{Conf}_k(\Gamma, \square) \simeq \textsc{Conf}_k(\Gamma, \boxslash)?$$

In both types of graphical configuration space, there are combinatorial methods for describing the cells of top dimension, referred to as *facets*.

Remark 4.3.8 (Facets in $\textsc{Conf}_k(\Gamma, \square)$). Let Γ be a graph and let k be a positive integer. The top-dimensional cells in $\textsc{Prod}_k(\Gamma, \square)$ are k-cubes, each of which can be labeled by an ordered list of k edges in Γ. If any pair of these edges shares a common vertex, then this labeled cube intersects the thick diagonal. Hence, the facets in $\textsc{Conf}_k(\Gamma, \square)$ are in one-to-one correspondence with the ways to choose an ordered list of k pairwise disjoint closed edges in Γ.

For the following characterization of cells in $\text{CONF}_k(\Gamma, \boxslash)$, we introduce some notation.

Remark 4.3.9 (Notation). If e is a directed edge in Γ, we introduce a slight abuse in notation and consider e as a function from $\{0, 1\}$ to the vertex set of Γ, where e begins at the vertex $e(0)$ and ends at $e(1)$. Two edges e_1 and e_2 are then disjoint if and only if $\{e_1(0), e_1(1)\} \cap \{e_2(0), e_2(1)\} = \emptyset$.

Remark 4.3.10 (Facets in $\text{CONF}_k(\Gamma, \boxslash)$). The facets in $\text{PROD}_k(\Gamma, \boxslash)$ are also labeled by ordered lists of k directed edges in Γ, but in this case the thick diagonal is slightly different. An ordered list $(e_1, \ldots, e_k)$ of directed edges in Γ labels a facet in $\text{CONF}_k(\Gamma, \boxslash)$ if, whenever $i < j$, then e_i and e_j are either disjoint or share a single vertex v such that $v = e_i(0) = e_j(1)$. The $k + 1$ vertices of this facet are then the following:

$$(e_1(0), e_2(0), \ldots, e_{k-1}(0), e_k(0))$$

$$(e_1(1), e_2(0), \ldots, e_{k-1}(0), e_k(0))$$

$$(e_1(1), e_2(1), \ldots, e_{k-1}(0), e_k(0))$$

$$\vdots$$

$$(e_1(1), e_2(1), \ldots, e_{k-1}(1), e_k(0))$$

$$(e_1(1), e_2(1), \ldots, e_{k-1}(1), e_k(1))$$

One should picture this as beginning with $(e_1(0), e_2(0), \ldots, e_{k-1}(0), e_k(0))$ and iteratively sliding vertices along edges based at those vertices.

When Γ consists of a single directed cycle, we obtain a natural example to apply the ideas above. In particular, this example is essential in later chapters.

Definition 4.3.11 (Oriented n-cycle)**.** The *oriented n-cycle* Γ_n is the directed graph on the vertex set $\mathbb{Z}/n\mathbb{Z}$ with n edges from i to $i+1$ for each $i \in \mathbb{Z}/n\mathbb{Z}$. If we consider $\mathbb{R}$ as a directed graph with vertex set $\mathbb{Z}$ and edges from i to $i+1$ for each $i \in \mathbb{Z}$, then Γ_n is the quotient of $\mathbb{R}$ by the action of $n\mathbb{Z}$ on $\mathbb{R}$ by translation. Similarly, for each positive integer $k < n$, the orthoscheme product $\text{PROD}_k(\Gamma_n, \boxslash)$ can be viewed as the quotient of the orthoscheme tiling for $\mathbb{R}^k$ by the action of $(n\mathbb{Z})^k$ by translation. By Remark 4.3.10, the facets in Γ_n are labeled by ordered lists of k edges in Γ_n with the prescribed restrictions. Furthermore, the vertices connected by an edge in $\text{CONF}_k(\Gamma_n, \boxslash)$ share the same cyclic ordering, and in general, motions of k points in Γ_n do not disturb the cyclic ordering of the vertices. Since there are $(k-1)!$ cyclic orderings of k vertices, there are then $(k-1)!$ connected components of the orthoscheme configuration space $\text{CONF}_k(\Gamma_n, \boxslash)$. For our purposes, we select the connected component which contains the vertex $(1, 2, \ldots, k) \in (\mathbb{Z}/n\mathbb{Z})^k$.

Example 4.3.12 (2 points in Γ_6)*.* As described in Example 4.3.2, the product $\text{PROD}_2(\Gamma_6, \boxslash)$ is a simplicial tiling of the torus into 72 triangles. The thick diagonal consists of 22 triangles which touch a $(1,1)$-curve on the torus. Removing these yields the (ordered) orthoscheme configuration space for 2 points in Γ_6, which is then homeomorphic to the direct product $\mathbb{S}^1 \times [0,1]$, i.e. an annulus. In Figure 4.2, each vertex is labeled by a configuration of two points in Γ_6, ordered by color. By gluing the bottom and top edges, we

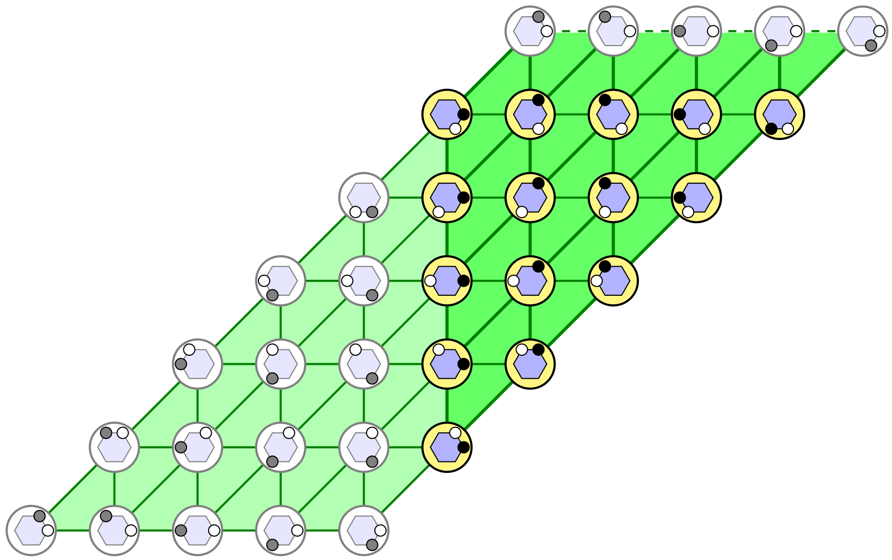

Figure 4.2: A portion of the universal cover for the configuration spaces of 2 points in Γ_6 - see Example 4.3.12.

obtain the orthoscheme structure for $\textsc{Conf}_2(\Gamma_6, \boxslash)$. The unordered counterpart, given by the quotient by $\textsc{Sym}_2$, is a Möbius band which is visualized in the figure by taking the subcomplex supported on the bolded vertices and gluing the left edge to the top (with greyed vertices). As illustrated in the figure, the universal cover of both configuration spaces is isometric to the $(2,6)$-dilated column - compare this with Example 3.5.8.

The main theorem of this chapter connects these configuration spaces with columns, our quintessential example of an orthoscheme complex. See [DMW] for more details.

Theorem 4.3.13. *The orthoscheme configuration space of k points in an oriented n-cycle is nonpositively curved and its* CAT(0) *universal cover is isometric to a (k,n)-dilated column.*

Proof. As remarked in Definition 4.3.11, $\text{PROD}_k(\Gamma_n, \boxslash)$ may be viewed as the quotient of the orthoscheme tiling for $\mathbb{R}^k$ by the subgroup of isometries given by translations in $(n\mathbb{Z})^k$. We can then see that each $(a_1, \ldots, a_k) \in \mathbb{Z}^k$ is sent via this quotient to a vertex of (our chosen connected component of) $\text{CONF}_k(\Gamma_n, \boxslash)$ if and only if there is a representative of $(a_1, \ldots, a_k)$ in $\mathbb{Z}^k/(n\mathbb{Z})^k)$ with distinct entries mod n. In other words, we can choose $b_1, \ldots, b_k$ such that

$$a_1 + b_1 n > a_2 + b_2 n > \cdots > a_k + b_k n > a_1 + (b_1 + 1)n.$$

The k-tuples $(a_1, \ldots, a_k)$ which satisfy this determine a subcomplex of the orthoscheme structure on $\mathbb{R}^k$ with infinitely many connected components, one for each choice of $(b_1, \ldots, b_k)$. For our conventions, we choose $b_i = 0$ for all i and observe that the resulting inequalities determine a (k, n)-dilated column by Definition 3.5.7. In other words, the universal cover of $\text{CONF}_k(\Gamma_n, \boxslash)$ is isometric to the (k, n)-dilated column and is thus CAT(0) by Proposition 3.5.9. $\square$

Example 4.3.14 (k points in Γ_n). The orthoscheme configuration space of k points in Γ_n is isometric to the metric direct product of Γ_n and a dilated Coxeter shape of type $\widetilde{A}_{n-1}$ where the dilation is by a factor of $n - k$. In particular, its universal cover is isometric to the (k, n)-dilated column.

It is worth noting that the definition of orthoscheme configuration space may be easily extended to include Δ-complexes and not just graphs. We introduce the following new type of configuration space.

Definition 4.3.15 (Orthoscheme Configuration Space)**.** Let Δ be a Δ-complex and let k be a positive integer. Then the *orthoscheme configuration space* $\textsc{Conf}_k(\Delta, \boxslash)$ of k points in Δ is the largest subcomplex of $\textsc{Prod}_k(\Delta, \boxslash)$ which does not contain the thick diagonal $\textsc{Diag}_n(\Delta^k)$. The corresponding *unordered configuration space* $\textsc{UConf}_k(\Delta, \boxslash)$ is given by the usual quotient by $\textsc{Sym}_k$.

As no work has yet been done on these configuration spaces, many natural questions are currently open.

Question 4.3.16. Let Δ be a nonpositively curved Δ-complex. Under what conditions is $\textsc{Conf}_k(\Delta, \boxslash)$ nonpositively curved? Is there a natural interpretation of the universal cover via dilated columns?

5. Noncrossing Partitions and the Dual Presentation

The study of noncrossing partitions originated from the work of Germain Kreweras in 1972 [Kre72] and has since spread to several other areas of mathematics, including the study of Coxeter groups. These combinatorial objects were connected to the symmetric group in the early 2000s and then generalized for other finite reflection groups [Bra01] [Arm09] [Rei97]. Our interest in noncrossing partitions is focused on their applications to the braid group in the form of an alternate group presentation and the construction of a related simplicial complex. In this chapter, we present the combinatorial background for noncrossing partitions, their applications to the braid group, and some newly defined subposets arising from this correspondence.

5.1 Noncrossing Partitions

We begin this chapter with a thorough discussion of noncrossing partitions and their uses. A *partition* of a finite set is a collection of pairwise disjoint subsets (called *blocks*) whose union is the entire set. The partitions of $[n] = \{1, \ldots, n\}$ form a poset Π_n under *refinement*: the partition σ is below the partition τ if each block of σ is contained in a block of τ. With this partial order, Π_n is referred to as the *partition lattice* of rank n.

Example 5.1.1 (Partition lattice). There are 5 partitions of [3] and 15 partitions of [4]; see Figures 5.1 and 5.2 for the corresponding Hasse diagrams.

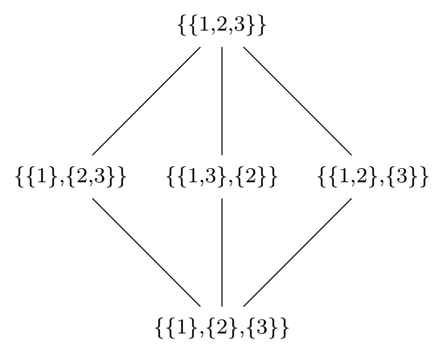

Figure 5.1: The Hasse diagram for Π_3, the partition lattice of rank 3

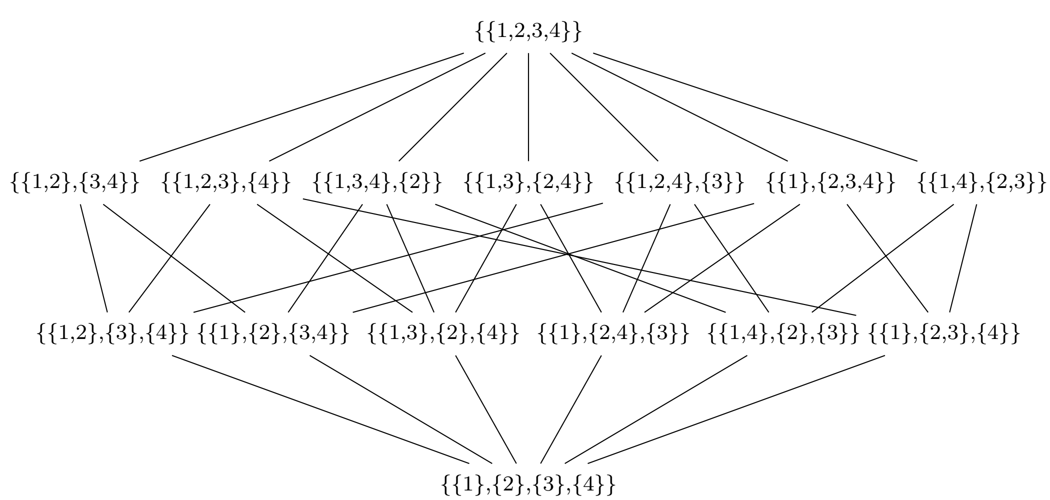

Figure 5.2: The Hasse diagram for Π_4, the partition lattice of rank 4

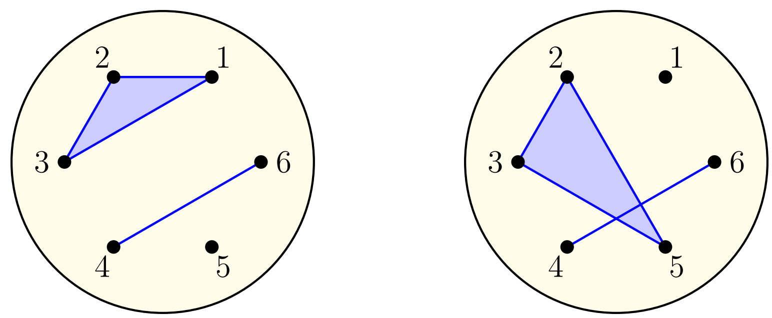

Figure 5.3: The noncrossing partition $\{\{1,2,3\},\{4,6\},\{5\}\}$ and the crossing partition $\{\{1\},\{2,3,5\},\{4,6\}\}$

In general, the partition lattice Π_n has size which is counted by the *Bell numbers* and is well-known in combinatorics for its many connections with other objects. However, our interest in partitions requires a restriction to a certain subposet.

Definition 5.1.2 (Noncrossing partitions)**.** Recall from Section 4.2 that each subset $A \subseteq [n]$ may be identified with a convex polygon in the following sense. The vertices of the regular n-gon D_n are given by $p_j = e^{2j\pi i/n}$ and each $A \subseteq [n]$ determines the vertex set $P_A = \{p_j \mid j \in A\}$; the convex hull $\textsc{Conv}(P_A)$ is then the convex polygon associated to A. Then each partition of $[n]$ may be identified with the collection of convex hulls of its blocks, and a partition is said to be *noncrossing* if these convex hulls are pairwise disjoint. We further declare a noncrossing partition to be *irreducible* if it contains exactly one non-singleton block. Refer to the set of all noncrossing partitions as NC_n and observe that this forms a subposet of Π_n - see Figure 5.4 for the Hasse diagram. In fact, NC_n and Π_n are both *lattices* in the sense that pairs of elements have unique meets and joins - see Definition 3.1.5. Both NC_n and Π_n are bounded and

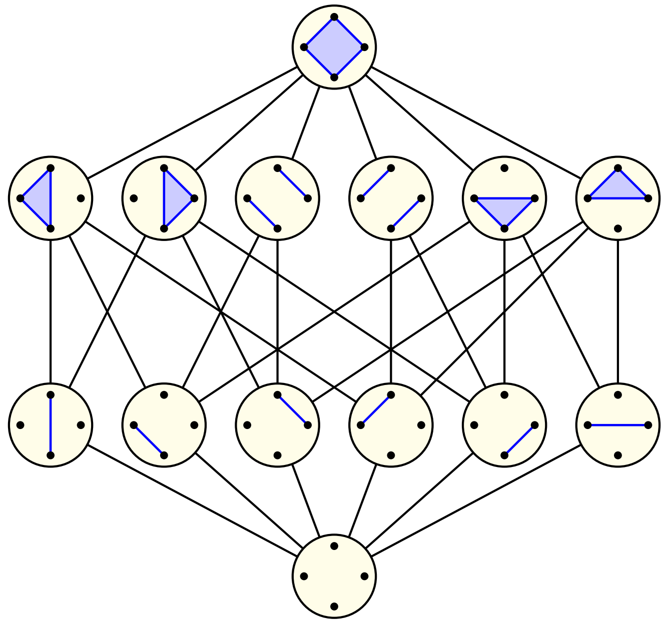

Figure 5.4: The noncrossing partition lattice NC_4

graded and we refer to the minimum and maximum elements as $\hat{0} = \{\{1\}, \ldots, \{n\}\}$ and $\hat{1} = \{[n]\}$ respectively. For each partition $\pi = \{A_1, \ldots, A_k\}$, the rank function is given by $\mathrm{rk}(\pi) = n - k$. Unlike Π_n, however, the noncrossing partition lattice is *self-dual*: it admits an order-reversing automorphism which can be described geometrically via the *Kreweras complement*, defined later in this section [Kre72]. Moreover, each interval $[\sigma, \tau] = \{\pi \in \mathrm{NC}_n \mid \sigma \le \pi \le \tau\}$ can be written as the direct product of smaller noncrossing partition lattices - see Remark 5.2.17.

Example 5.1.3. Every partition of [3] is noncrossing, so $\Pi_3 = \mathrm{NC}_3$. The partition $\{\{1,3\},\{2,4\}\}$ is the unique crossing element of Π_4; there are 14 elements in NC_4, displayed in Figure 5.4.

Remark 5.1.4 (Meets and joins). For all noncrossing partitions $\sigma, \tau \in \mathrm{NC}_n$, the meet and join are easy to describe in the refinement order. The meet $\sigma \wedge \tau$ is the coarsest common refinement of the two partitions, i.e.

$$\sigma \wedge \tau = \{A \cap B \mid A \in \sigma, B \in \tau\}.$$

Similarly, the join $\sigma \vee \tau$ is the finest noncrossing partition which has both σ and τ as a refinement. Notice that, while $\sigma \wedge \tau$ has identical definitions in both NC_n and Π_n, the same is not true for $\sigma \vee \tau$. For example, if we let $\sigma = \{\{1,3\},\{2\},\{4\}\}$ and $\tau = \{\{1\},\{2,4\},\{3\}\}$, then $\sigma \vee \tau$ in Π_4 is the partition $\{\{1,3\},\{2,4\}\}$, but since this is a crossing partition, $\sigma \vee \tau$ in NC_4 is equal to $\{\{1,2,3,4\}\}$.

The lattice of noncrossing partitions is the home of several nice counting problems. The number of elements in NC_n is the *Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

an integer sequence famous in combinatorics for its long list of appearances [Kre72] [McC06] [Sta99]. Additionally, the number of maximal chains (e.g. paths from minimum to maximum in the Hasse diagram) in NC_n is n^{n-2} [Ede80], a number associated to counting many combinatorial objects, including binary rooted trees, labeled Dyck paths, labeled rooted forests, and parking functions.

5.2 Dual Simple Braids

Beyond their combinatorial connections, noncrossing partitions have a useful group-theoretic interpretation. Each noncrossing partition corresponds to a braid (and thus, a

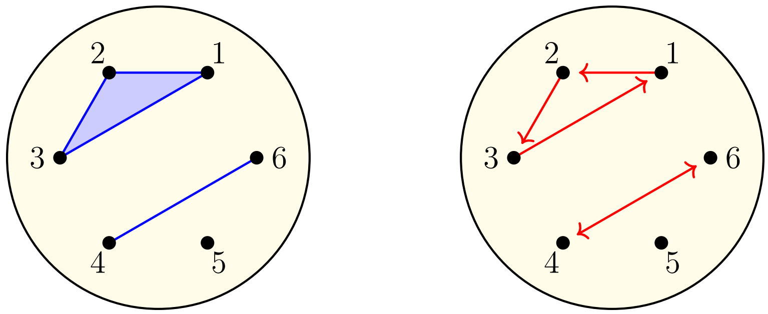

Figure 5.5: The noncrossing partition $\{\{1,2,3\},\{4,6\},\{5\}\}$ and the corresponding noncrossing permutation $(1\ 2\ 3)(4\ 6)(5)$

permutation) obtained via a motion of points in the unit disk determined by the partition. Understanding these braids is essential to our perspective for the braid group.

Definition 5.2.1 (Dual simple braids)**.** For each noncrossing partition $\pi = \{A_1, \ldots, A_k\}$ in NC_n, define the *dual simple braid* δ_π by the product of rotations (Definition 4.2.13) $\delta_{A_1} \cdots \delta_{A_k}$ and observe that the order of multiplication is unimportant since π is noncrossing. Notice further that the $\binom{n}{2}$ braids corresponding to partitions which cover the minimum element $\hat{0}$ are precisely the positive half-twists. When π has a unique non-singleton block A, then δ_π is simply the rotation braid δ_A. In the special case of $A = [n]$, we denote $\delta_{[n]}$ by δ_n. This provides an injection $\mathrm{NC}_n \to \mathrm{B{\scriptsize RAID}}_n$, and via the map $\mathrm{B{\scriptsize RAID}}_n \to \mathrm{S{\scriptsize YM}}_n$ given by following strands (see Section 2.1), we also obtain an injection $\mathrm{NC}_n \to \mathrm{S{\scriptsize YM}}_n$ which we denote as sending π to σ_π. Hence, the noncrossing partition π is associated to both a canonical dual simple braid δ_π and a canonical *noncrossing permutation* σ_π. We refer to the set of dual simple braids as DS_n and the set of noncrossing permutations as NP_n and remark that each inherits a poset structure with is isomorphic to that of NC_n. Indeed, there is no combinatorial distinction between noncrossing partitions, noncrossing permutations, and dual simple braids. We will shift between these three settings as is convenient, typically using the symbol π to represent a noncrossing partition, with σ_π and δ_π representing the associated noncrossing permutation and dual simple braid, respectively.

Although the set of positive half-twists is closed under conjugation by dual simple braids, the conjugation of a dual simple braid is not, in general, another dual simple braid. However, there are certain situations in which this property is preserved.

Proposition 5.2.2. *If $\sigma, \tau \in \mathrm{NC}_n$ and $\sigma \leq \tau$, then ${\delta_\tau}^{-1}\delta_\sigma\delta_\tau$ is a dual simple braid.*

Proof. First, consider the case when $\tau = \{[n]\}$. Recall that δ_n is represented by the counter-clockwise rotation of the vertex set P_n by an angle of $2\pi/n$, thus sending each vertex to the next in cyclic order. Hence, conjugating by δ_n preserves the block structure of δ_π, but with shifted vertex labels. In other words, ${\delta_n}^{-1}\delta_\pi\delta_n$ is a dual simple braid, corresponding to the noncrossing partition obtained via rotation of the diagram for π through an angle of $2\pi/n$.

The above result immediately implies the following: if $A \subseteq B \subseteq [n]$, then ${\delta_B}^{-1}\delta_A\delta_B$ is a rotation braid. We may see this by restricting to the dual simple braids which lie below δ_B in the refinement order, i.e. those which may be described as motions on the subdisk D_B. We discuss braids of this type further in Definition 5.2.13.

Now, for the generic case $\sigma \leq \tau$, the blocks of σ are obtained as refinements of the blocks in τ, and thus ${\delta_\tau}^{-1}\delta_\sigma\delta_\tau$ is written as the product of terms of the form ${\delta_B}^{-1}\delta_A\delta_B$, where $A \in \sigma$ and $B \in \tau$. Thus ${\delta_\tau}^{-1}\delta_\sigma\delta_\tau$ is the product of disjoint rotation braids and hence a dual simple braid. □

Associating noncrossing partitions to elements of the braid group endows these braids with a partial order, but not an unfamiliar one. In particular, the map $\mathrm{NC}_n \to \textsc{Braid}_n$ given above gives an embedding of the noncrossing partition lattice into a Cayley graph for the braid group.

Definition 5.2.3 (Cayley Graph)**.** Let G be a group with fixed generating set S. Then the *(right) Cayley graph of G with respect to S* is the directed graph whose vertex set is indexed by G such that, for each $g_1, g_2 \in G$, there is a directed edge e_{g_1,g_2} from v_{g_1} to v_{g_2} if and only if there exists $s \in S$ such that $g_1 s = g_2$. Denote this graph by $\mathrm{Cay(G,S)}$. Then G acts freely and transitively on $\mathrm{Cay(G,S)}$ by labeled graph isomorphism via left multiplication on the vertex labels.

Recall from Definition 4.2.13 that the positive half-twists in $\textsc{Braid}_n$ are the $\binom{n}{2}$ elements obtained by rotations δ_e, where $e = \{i,j\} \subseteq [n]$.

Proposition 5.2.4. *Let T be the set of all positive half-twists in $\textsc{Braid}_n$. Then the map $\pi \mapsto \delta_\pi$ which sends each noncrossing partition to its corresponding dual simple braid gives an embedding of the Hasse diagram for NC_n into the (right) Cayley graph for $\textsc{Braid}_n$ with respect to the generating set T.*

Proof. It suffices to show that for each covering relation $\sigma < \tau$ in NC_n, the corresponding dual simple braids in DS_n satisfy the property that ${\delta_\tau}^{-1}\delta_\sigma$ is a positive half-twist. Since

$\sigma < \tau$ is a covering relation, τ may be obtained by replacing two blocks $A, B \in \sigma$ with the union $A \cup B$. Then by rearranging commuting terms, we have ${\delta_\sigma}^{-1}\delta_\tau = {\delta_A}^{-1}{\delta_B}^{-1}\delta_{A\cup B}$, so it suffices to show that elements of this form are positive half-twists.

That is, let A and B be disjoint subsets of $[n]$ such that $\textsc{Conv}(P_A) \cap \textsc{Conv}(P_B) = \emptyset$. Then we may write $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_l\}$ such that, in the cyclic order corresponding to reading labels on the n-gon D_n counter-clockwise, the elements of A and B appear in the order $a_1, \ldots, a_k, b_1, \ldots, b_l$. Then δ_A and δ_B are rotation braids which correspond to the noncrossing permutations $(a_1 \ \ldots \ a_k)$ and $(b_1 \ \ldots \ b_l)$, respectively. Furthermore, we may observe that

$$(a_1 \ \ldots \ a_k)(b_1 \ \ldots \ b_l)(a_1 \ b_1) = (a_1 \ \ldots \ a_k \ b_1 \ \ldots \ b_l)$$

and similarly, we have $\delta_A\delta_B\delta_{\{a_1,b_1\}} = \delta_{A\cup B}$, as desired.

Therefore, the Hasse diagram for NC_n embeds into the Cayley graph for $\textsc{Braid}_n$ with respect to T. □

By following the natural map from $\textsc{Braid}_n$ to $\textsc{Sym}_n$, we may similarly embed the Hasse diagram for NC_n into the Cayley graph for $\textsc{Sym}_n$ with respect to the generating set of all transpositions. Moreover, the images of these embeddings have a particularly nice structure. The following theorem may be found in [Arm09] and [Bia97] for the $\textsc{Sym}_n$ case, but is easily extended to $\textsc{Braid}_n$.

Theorem 5.2.5 (Geodesic Intervals)**.** *The Hasse diagram for* NC_n *is isomorphic to the subgraph of the Cayley graph* $\mathrm{Cay}(\textsc{Braid}_n, T)$ *determined by all geodesic (length-minimizing) paths from the identity braid e to δ_n.*

Embedding the Hasse diagram for NC_n into a Cayley graph allows us to interpret combinatorial data in group-theoretic terms. Among these is the following generalization of the rank function for NC_n.

Definition 5.2.6 (Absolute Reflection Length)**.** Let $R = T \cup T^{-1}$ denote the set of all positive half-twists and their inverses. For each $\beta \in \textsc{Braid}_n$, let $\ell(\beta)$ be the smallest integer k such that $\beta = \beta_1 \cdots \beta_k$ and each $\beta_i \in R$. In the case of the identity braid e, we define $\ell(e) = 0$. We refer to ℓ as the *absolute reflection length* on $\textsc{Braid}_n$ and observe that for each $\pi \in \mathrm{NC}_n$, the dual simple braid δ_π has absolute reflection length $\ell(\delta_\pi) = \mathrm{rk}(\pi)$. Notice further that for all $\beta_1, \beta_2 \in \textsc{Braid}_n$, we have $\ell(\beta_1\beta_2) \leq \ell(\beta_1) + \ell(\beta_2)$. Finally, we note that since edges in the Cayley graph for $\textsc{Braid}_n$ are labeled by the positive half-twists, then any directed geodesic path from β_1 to β_2 has length $\ell({\beta_1}^{-1}\beta_2)$.

Again, we may similarly define the absolute reflection length on $\textsc{Sym}_n$ by recording the minimum number of transpositions needed to write a given permutation as a product. In [McC], McCammond provides a useful characterization of dual simple braids which arises from the absolute reflection length.

Theorem 5.2.7 ([McC], Proposition 6.6)**.** *If $\beta = \beta_1 \cdots \beta_k$ in $\textsc{Braid}_n$ and we have $\ell(\beta) = \ell(\beta_1) + \cdots + \ell(\beta_k)$, then β is a dual simple braid if and only if each β_i is a dual simple braid.*

The embedding from Theorem 5.2.5 also grants a useful interpretation for the refinement order on NC_n.

Theorem 5.2.8. *If $\sigma, \tau \in \mathrm{NC}_n$, then $\sigma \leq \tau$ if and only if ${\delta_\sigma}^{-1}\delta_\tau$ is a dual simple braid.*

Proof. Suppose $\sigma \leq \tau$ in NC_n. By Theorem 5.2.5, $\sigma \leq \tau$ if and only if δ_σ and δ_τ lie on a geodesic path from e to δ_n in $\mathrm{Cay}(\textsc{Braid}_n, T)$. Then the labels on this path are positive half-twists $\beta_1, \ldots, \beta_{n-1}$ such that

$$\beta_1 \cdots \beta_i = \delta_\sigma$$

$$\beta_1 \cdots \beta_j = \delta_\tau$$

$$\beta_1 \cdots \beta_{n-1} = \delta_n$$

where $i = \mathrm{rk}(\sigma) \leq \mathrm{rk}(\tau) = j$. In particular, observe that $\beta_{i+1} \cdots \beta_{n-1} = {\delta_\sigma}^{-1}\delta_n$ and $\beta_{i+1} \cdots \beta_j = {\delta_\sigma}^{-1}\delta_\tau$.

By Lemma 5.2.2, the braid ${\delta_n}^{-1}\delta_\sigma\delta_n$ is a dual simple braid with the same rank as δ_σ; hence, there are positive half-twists $\gamma_1, \ldots, \gamma_i$ such that $\gamma_1 \cdots \gamma_i = {\delta_n}^{-1}\delta_\sigma\delta_n$. Therefore,

$$\begin{aligned}\beta_{i+1} \cdots \beta_{n-1}\gamma_1 \cdots \gamma_i &= ({\delta_\sigma}^{-1}\delta_n)({\delta_n}^{-1}\delta_\sigma\delta_n) \\ &= \delta_n\end{aligned}$$

and thus the positive half-twists $\beta_{i+1}, \ldots, \beta_{n-1}, \gamma_1, \ldots, \gamma_i$ label a geodesic path from e to δ_n in the Cayley graph for $\textsc{Braid}_n$ with respect to T. By Theorem 5.2.5, every vertex on this path labels a dual simple braid, so we may conclude that $\beta_{i+1} \cdots \beta_j = {\delta_\sigma}^{-1}\delta_\tau$ is a dual simple braid, as desired.

Similarly, suppose ${\delta_\sigma}^{-1}\delta_\tau$ is a dual simple braid. Bu Theorem 5.2.5, ${\delta_\sigma}^{-1}\delta_\tau$ then labels a vertex on a geodesic path in the Cayley graph from the identity to δ_n. Since δ_σ and

δ_τ are also dual simple braids, another application of Theorem 5.2.5 tells us that they also lie on geodesics from the identity to δ_n. Then the same labels for the path from e to ${\delta_\sigma}^{-1}\delta_\tau$ label a geodesic path from δ_σ to δ_τ, and hence the two lie on a common geodesic from e to δ_n and $\sigma \leq \tau$. □

The theorem above tells us that if $\delta_\sigma \leq \delta_\tau$ in DS_n, then the label on the edge from δ_σ to δ_τ in the Hasse diagram is also a dual simple braid. We may additionally conclude that $\delta_\tau{\delta_\sigma}^{-1}$ is a dual simple braid.

Proposition 5.2.9. *If $\sigma, \tau \in \mathrm{NC}_n$, then ${\delta_\sigma}^{-1}\delta_\tau \in \mathrm{DS}_n$ if and only if $\delta_\tau{\delta_\sigma}^{-1} \in \mathrm{DS}_n$.*

Proof. It is straightforward to see that the absolute reflection lengths $\ell({\delta_\sigma}^{-1}\delta_\tau)$ and $\ell(\delta_\tau{\delta_\sigma}^{-1})$ are equal. Then

$$\ell(\delta_\sigma) + \ell({\delta_\sigma}^{-1}\delta_\tau) = \ell(\delta_\tau{\delta_\sigma}^{-1}) + \ell(\delta_\sigma)$$

and thus by Theorem 5.2.7, ${\delta_\sigma}^{-1}\delta_\tau$ is a dual simple braid if and only if $\delta_\tau{\delta_\sigma}^{-1}$ is a dual simple braid. □

A particularly useful example of Theorem 5.2.8 is found when $A \subseteq B \subseteq [n]$, in which case $\delta_A \leq \delta_B$ and the braid ${\delta_A}^{-1}\delta_B$ is straightforward to understand.

Remark 5.2.10 (Rotation Braids). Let $A \subseteq B \subseteq [n]$ and let $A = \{a_1, \ldots, a_k\}$ with the ordering $a_1 < \cdots < a_k$. Then for each $i \in [k]$, let $B_i = \{b_{i_1}, \ldots, b_{i_j}\}$ be the (possibly empty) set of linearly ordered elements in B which appear between a_i and a_{i+1} (or a_k and a_1 when $i = k$). We may then define $A_i = B_i \cup \{a_i\}$ for each i and observe that

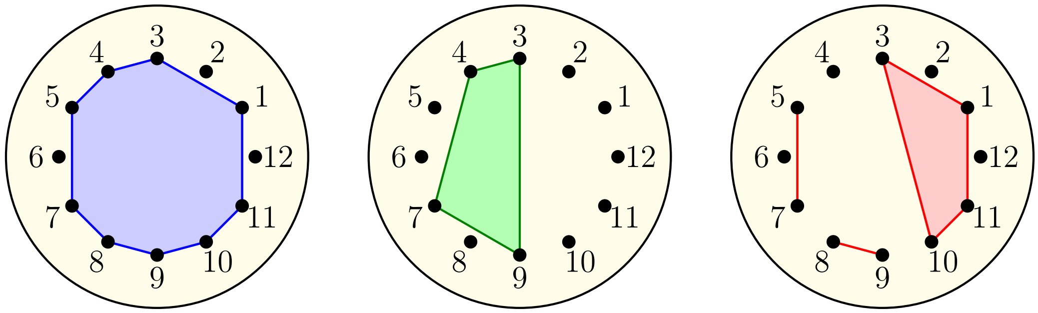

Figure 5.6: The braids δ_B, δ_A, and ${\delta_A}^{-1}\delta_B$ in NC_{12} when $A = \{3,4,7,9\}$ and $B = \{1,3,4,5,7,8,9,10,11\}$

$\{A_1, \ldots, A_k\}$ is a noncrossing partition of B. We may then include each remaining element of $[n]$ as a singleton to get a noncrossing partition of $[n]$. In other words,

$$\pi = \{A_1, \ldots, A_k\} \cup \{\{i\} \mid i \in [n] - B\}$$

is a noncrossing partition of $[n]$. Furthermore, we can see that

$$\mathrm{rk}(\delta_\pi) = \sum_{i=1}^{k} |A_i| - 1 = |B| - |A|$$

and $\delta_A \delta_\pi = \delta_B$.

Example 5.2.11. If $A = \{3,4,7,9\}$, $B = \{1,3,4,5,7,8,9,10,11\}$, and $n = 12$, then ${\delta_A}^{-1}\delta_B$ is the product of rotation braids $\delta_{\{1,3,10,11\}}\delta_{\{5,7\}}\delta_{\{8,9\}}$ - see Figure 5.6. Overlaying the three diagrams as seen in Figure 5.7 lends some geometric intuition for how we find ${\delta_A}^{-1}\delta_B$ in general.

Example 5.2.12. When $B = [n]$, the braids above are easier to describe. In this case, let $A = \{a_1, \ldots, a_k\}$ with $a_1 < \cdots < a_k$ as before. Then define A_i to be the largest (finite) set of the form $\{a_i, a_i + 1, a_i + 2, \ldots\}$ (reduced mod n) such that $a_{i+1} \notin A_i$ for

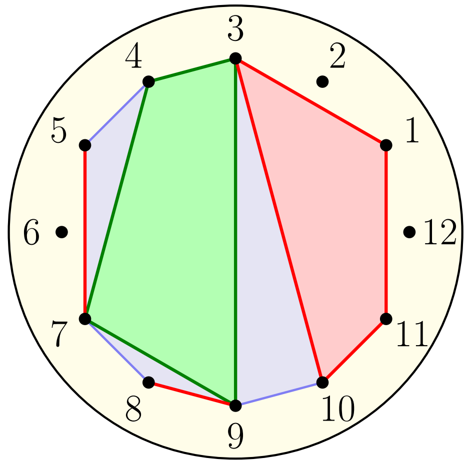

Figure 5.7: The braids from Example 5.2.11, overlaid

all $i \in [k-1]$ and $a_1 \notin A_k$. Then $\pi = \{A_1, \ldots, A_k\}$ is a noncrossing partition of $[n]$ and ${\delta_A}^{-1}\delta_n = \delta_\pi$.

Dual simple braids provide a useful way of visualizing the structure of intervals in the noncrossing partition lattice. We begin by considering intervals of the form $[\hat{0}, \delta_A]$ for each $A \subseteq [n]$, where $\hat{0}$ in this context denotes the trivial braid. As we will see, understanding this case is sufficient for understanding generic intervals of NC_n.

Definition 5.2.13 (Sublattices)**.** Let $A \subseteq [n]$ and define the subposet DS_A of DS_n to be the interval $[\hat{0}, \delta_A]$, where δ_A is the rotation braid on the vertex set P_A. Define NC_A and NP_A to be the corresponding subposets of NC_n and NP_n, respectively. One can easily see that if $|A| = k$, then NC_A is isomorphic to the noncrossing partition lattice NC_k by identifying A with $[k]$.

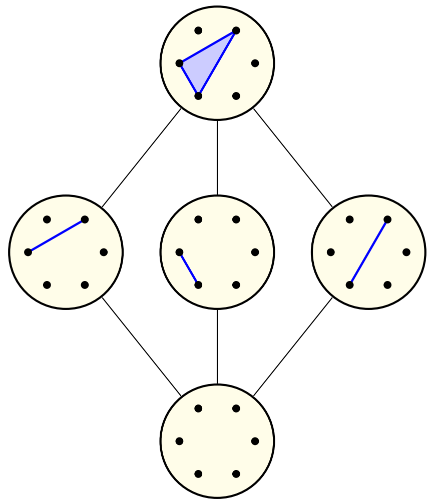

Figure 5.8: The sublattice $\mathrm{NC}_{\{1,3,4\}} \cong \mathrm{NC}_3$ in NC_6

Remark 5.2.14 (Notation). While we obscure reference to n in the notation NC_A, we consider the ambient set $[n]$ containing A to be a distinguishing feature. That is, if $m \neq n$ with $A \subseteq [n]$ and $A' \subseteq [m]$ with $A = A'$, then $\mathrm{NC}_A \neq \mathrm{NC}_{A'}$, although the two lattices are isomorphic. To avoid confusion, we generally refer to NC_A when n is fixed.

Example 5.2.15. Consider the subset $A = \{1, 3, 4\}$ of $[6]$. Then NC_A is isomorphic to NC_3, as depicted in Figure 5.8.

The intersections of these sublattices are easy to describe.

Proposition 5.2.16 (Sublattice Intersections). *Let A_1 and A_2 be subsets of $[n]$. Then $\mathrm{NC}_{A_1} \cap \mathrm{NC}_{A_2} = \mathrm{NC}_{A_1 \cap A_2}$.*

Proof. Since $A_1 \cap A_2 \subseteq A_1$ and $A_1 \cap A_2 \subseteq A_2$, we have that $\delta_{A_1 \cap A_2} \leq \delta_{A_1}$ and $\delta_{A_1 \cap A_2} \leq \delta_{A_2}$ in the partial order on dual simple braids. Then since $\mathrm{DS}_{A_1 \cap A_2} = [\hat{0}, \delta_{A_1 \cap A_2}]$, we know that this is a subposet of both DS_{A_1} and DS_{A_2}, and we have $\mathrm{DS}_{A_1\cap A_2} \subseteq \mathrm{DS}_{A_1} \cap \mathrm{DS}_{A_2}$. Following our natural isomorphisms, we further have $\mathrm{NC}_{A_1\cap A_2} \subseteq \mathrm{NC}_{A_1} \cap \mathrm{NC}_{A_2}$.

The other direction follows immediately from the observation that the meet of δ_{A_1} and δ_{A_2} is $\delta_{A_1} \wedge \delta_{A_2} = \delta_{A_1\cap A_2}$. To see this, let π_1 and π_2 be noncrossing partitions in NC_n with the property that A_1 and A_2 are blocks in π_1 and π_2 respectively, and all other blocks in each are singletons. Then the coarsest common refinement of π_1 and π_2 has $A_1 \cap A_2$ as a block with all others singletons. □

As it turns out, every interval in NC_n is isomorphic to a product of the sublattices described above.

Remark 5.2.17 (Intervals in NC_n). Let $\pi \in \mathrm{NC}_n$ and consider the interval $[\hat{0}, \pi]$. For each $\tau \in [\hat{0}, \pi]$, the blocks of π may be obtained via unions of blocks of τ. In other words, τ is determined by a choice of partition for each block in π. If $\pi = \{A_1, \ldots, A_k\}$, then this gives a map

$$[\hat{0}, \pi] \to \mathrm{NC}_{A_1} \times \cdots \times \mathrm{NC}_{A_k}$$

per Definition 5.2.13 and it is straightforward to show that this map is an order-preserving bijection. In other words, the interval $[\hat{0}, \pi]$ is isomorphic to a product of smaller noncrossing partition lattices, with sizes depending on the sizes of the blocks in π. More generally, each interval of the form $[\pi_1, \pi_2]$ is isomorphic to an interval of the form $[\hat{0}, \pi]$, where $\pi_1 \leq \pi_2$ in NC_n. To see this, let $\tau \in [\pi_1, \pi_2]$; then $\delta_{\pi_1}^{-1}\delta_\tau$ is a dual simple braid, and the corresponding noncrossing partition lies in the interval $[\hat{0}, \pi]$. This produces an order-preserving bijection, and hence every interval in NC_n is isomorphic to a product

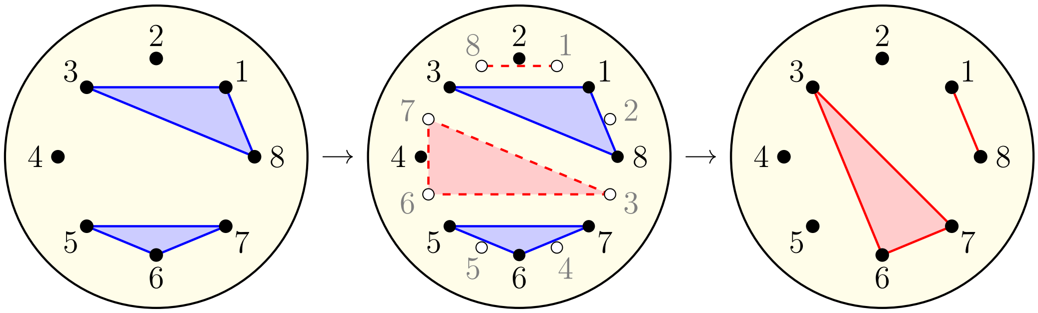

Figure 5.9: The Kreweras complement

of smaller noncrossing partition lattices. While Kreweras adopted a more combinatorial approach to his exposition, the use of dual simple braids is indicative of the perspective we use in this setting.

To see the utility granted by embedding NC_n in a group, consider the following anti-automorphism for the noncrossing partition lattice.

Definition 5.2.18 (Kreweras Complement)**.** For each $\delta_\pi \in \mathrm{DS}_n$, define $K(\delta_\pi)$ to be the dual simple braid $\delta_\pi^{-1}\delta_n$. Then K defines an anti-automorphism of DS_n (and thus NC_n) referred to as the *Kreweras complement*. This map then induces an anti-automorphism on products of noncrossing partition lattices by performing K on each component and hence, by Remark 5.2.17, we have an anti-automorphism on each interval in DS_n. The resulting map on $[\hat{0}, \delta_\pi]$ is then called the *relative Kreweras complement* with respect to δ_π. Each of these maps plays a useful role in the poset structure of NC_n - see [NS96] for a survey.

The above is actually not quite the definition given by Kreweras, but it is equivalent up to a symmetry of the n-gon. The original definition for the Kreweras complement given in [Kre72] was explicitly combinatorial, with no reference to group structure. Instead, Kreweras describes the following procedure. Define $\overline{p}_k = e^{\pi i(2k+1)/2n}$ and consider the additional n vertices $\overline{P}_n = \{\overline{p}_k \mid k \in \mathbb{Z}\}$ in $\mathbb{D}^2$. Then for each $\pi \in \mathrm{NC}_n$, there is a unique largest $K(\pi) \in \mathrm{NC}_n$ with the property that when the blocks of π and $K(\pi)$ are drawn via convex hulls on vertex sets P_n and $\overline{P}_n$ respectively, the resulting blocks are all pairwise noncrossing. See Figure 5.2 for an example.

5.3 Factorizations

As exhibited in the preceding section, many properties of the poset structure for NC_n can be usefully rephrased using the group structure from Braid_n. In this section, we examine the ways in which we can write a fixed dual simple braid as a product with certain restrictions on the factors.

Recall that if $\sigma < \tau$ is a covering relation in NC_n, then the product ${\delta_\sigma}^{-1}\delta_\tau$ is a positive half-twist (Theorem 5.2.8), and this half-twist labels the edge between vertices labeled by δ_σ and δ_τ in the Cayley graph for Braid_n. As described in Theorem 5.2.5, the maximal chains of NC_n are in bijection with the geodesic paths from the identity to δ_n, each of which yields a factorization of δ_n into positive half-twists. More generally, we have the following definition.

Definition 5.3.1 (Partial Factorizations)**.** A *factorization* of δ_n is an $(n-1)$-tuple of positive half-twists $(\delta_{e_1}, \ldots, \delta_{e_{n-1}})$ with the property that $\delta_{e_1} \cdots \delta_{e_{n-1}} = \delta_n$. More generally, a *partial factorization* (or *k-factorization*) of the dual simple braid $\delta_\pi \in \mathrm{DS}_n$ is a k-tuple of dual simple braids $(\delta_{\pi_1}, \ldots, \delta_{\pi_k})$ such that $\delta_{\pi_1} \cdots \delta_{\pi_k} = \delta_\pi$ and $\mathrm{rk}(\pi_1)+\cdots+\mathrm{rk}(\pi_k) = \mathrm{rk}(\sigma)$. Following the results of Section 5.2, observe that $\hat{0} = \pi_0 < \pi_1 < \cdots < \pi_{k-1} < \pi_k = \hat{1}$ is a k-chain in NC_n if and only if the k-tuple of dual simple braids $(\delta_{\tau_1}, \delta_{\tau_2}, \ldots, \delta_{\tau_k})$ is a partial k-factorization of δ_n, where $\delta_{\pi_{i-1}} \delta_{\tau_i} = \delta_{\pi_i}$ for each i. Finally, we observe that this imposes a partial order on the set of all partial factorizations for δ_n: one partial factorization sits below another if and only if the chain in NC_n which corresponds to the former is a subchain of the latter.

By Proposition 5.2.7, the restriction to considering dual simple braids in partial factorizations is not a restriction at all. To make use of the structure for partial factorizations, we introduce the following combinatorial condition on dual simple braids.

Definition 5.3.2 (Properly ordered)**.** If A and B are subsets of $[n]$, then the ordered pair (A, B) is *properly ordered* if the corresponding convex hulls $\textsc{Conv}(P_A)$ and $\textsc{Conv}(P_B)$ are either noncrossing (i.e. disjoint) or they intersect in the single vertex p_i with the property that the sequence $p_{i+1}, p_{i+2}, \ldots$ encounters all elements of $P_A \backslash \{p_i\}$ before any element of P_B. This last condition may be verified geometrically; if we consider the vertex p_i, then "looking in" at D_n right-to-left from p_i (i.e. counter-clockwise) reveals $\textsc{Conv}(P_A)$ before $\textsc{Conv}(P_B)$. Similarly, if π_1 and π_2 are noncrossing partitions of $[n]$, then the ordered pair (π_1, π_2) is *properly ordered* if for each pair of blocks $A_1 \in \pi_1$ and

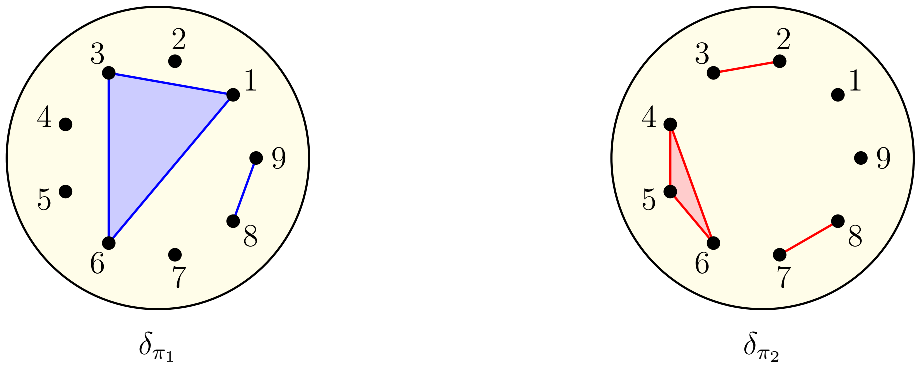

Figure 5.10: Dual simple braids δ_{π_1} and δ_{π_2} such that $(\delta_{\pi_1}, \delta_{\pi_2})$ are properly ordered - see Example 5.10

$A_2 \in \pi_2$, we have that (A_1, A_2) is properly ordered. By the geometric comment above, proper ordering of partitions may be verified locally for each vertex of D_n.

Example 5.3.3. Consider the partitions $\pi_1 = \{\{1,3,6\},\{2\},\{4\},\{5\},\{7\},\{8,9\}\}$ and $\pi_2 = \{\{1\},\{2,3\},\{4,5,6\},\{7,8\},\{9\}\}$. Then $(\delta_{\pi_1}, \delta_{\pi_2})$ is properly ordered, as seen in Figure 5.10. By Proposition 5.3.4, the product $\delta_{\pi_1}\delta_{\pi_2}$ is a dual simple braid - one can easily check that the result is δ_π, where $\pi = \{\{1,2,3,4,5,6\},\{7,8,9\}\}$.

The following proposition may be found in [McC], where it is used to prove several properties for partial factorizations.

Proposition 5.3.4 ([McC], Lemma 7.1)**.** *If $\sigma, \tau \in \mathrm{NC}_n$, then (σ, τ) is properly ordered if and only if $\delta_\sigma \delta_\tau = \delta_{\sigma \vee \tau}$. Moreover, if $(\delta_{\pi_1}, \ldots, \delta_{\pi_k})$ is a partial factorization for δ_n, then (π_i, π_j) is properly ordered whenever $i < j$.*

Before continuing on, we pause to mention the following immediate corollary.

Proposition 5.3.5. *If $\sigma, \tau \in \mathrm{NC}_n$ and (σ, τ) is properly ordered, then ${\delta_\tau}^{-1}\delta_\sigma\delta_\tau$ and $\delta_\tau\delta_\sigma{\delta_\tau}^{-1}$ are dual simple braids.*

Proof. By Proposition 5.3.4, ${\delta_\tau}^{-1}\delta_\sigma\delta_\tau = {\delta_\tau}^{-1}\delta_{\sigma\vee\tau}$. Since $\tau \leq \sigma \vee \tau$, applying Theorem 5.2.8 tells us that ${\delta_\tau}^{-1}\delta_\sigma\delta_\tau \in \mathrm{DS}_n$ and thus by Proposition 5.2.9 $\delta_\tau\delta_\sigma{\delta_\tau}^{-1} \in \mathrm{DS}_n$ as well. □

Using the propositions above, we now describe a method for transforming one partial factorization of δ_n into another. Although this procedure is far more general than our application, we describe this case specifically for the braid group.

Definition 5.3.6 (Hurwitz Moves)**.** Let $f = (\delta_{e_1}, \ldots, \delta_{e_{n-1}})$ be a factorization of δ_n, where each δ_{e_i} is a positive half-twist. For each $i \in [n-2]$, define the $(n-1)$-tuple

$$h_i(f) = (\delta_{e_1}, \ldots, \delta_{e_{i-1}}, \delta_{e_{i+1}}^{\delta_{e_i}}, \delta_{e_i}, \delta_{e_{i+2}}, \ldots, \delta_{e_{n-1}}),$$

where we recall that a^b denotes the conjugation bab^{-1} and notice that $h_i(f)$ is also a factorization of δ_n. Similarly, the inverse of this map is written

$$h_i^{-1}(f) = (\delta_{e_1}, \ldots, \delta_{e_{i-1}}, \delta_{e_{i+1}}, \delta_{e_i}^{\delta_{e_{i+1}}^{-1}}, \delta_{e_{i+2}}, \ldots, \delta_{e_{n-1}})$$

and also yields another factorization by Propositions 5.3.4 and 5.3.5. Each h_i and its inverse is referred to as a *Hurwitz move*. More generally, if $p = (\delta_{\pi_1}, \ldots, \delta_{\pi_k})$ is a k-factorization of δ_n, then we may perform a Hurwitz move on this partial factorization by declaring

$$h_i(p) = (\delta_{\pi_1}, \ldots, \delta_{\pi_{i-1}}, \delta_{\pi_{i+1}}^{\delta_{\pi_i}}, \delta_{\pi_i}, \delta_{\pi_{i+2}}, \ldots, \delta_{\pi_k}).$$

If $A \subseteq B \subseteq [n]$, then each partial factorization of δ_A may be extended into one of δ_B. We begin by describing the case when $B = [n]$ and $A = [n] - \{s\}$ for some $s \in [n]$. First, for each $s \in [n]$, let δ_{e_s} be the positive half-twist corresponding to the transposition $(s-1\ s)$ if $s > 1$ and $(1\ n)$ if $s = 1$.

Remark 5.3.7 (Extending Factorizations of δ_n). Let $s \in [n]$ and define $A = [n] - \{s\}$ and δ_{e_s} as above. Then each partial factorization $p = (\delta_{\pi_1}, \ldots, \delta_{\pi_{k-1}})$ of δ_A can be extended into k different partial factorizations of δ_n via the following procedure. First, define

$$p' = (\delta_{\pi_1}, \ldots, \delta_{\pi_{k-1}}, \delta_{e_s})$$

and observe that p' is a k-factorization of δ_n. From here, we may iteratively apply Hurwitz moves to move δ_{e_s} to any of the $k-1$ other positions, each yielding a new factorization of δ_n. See Example 5.3.10. As we later describe, this process may be iterated, providing a useful way to produce factorizations of δ_n from factorizations of a smaller rotation braids.

Remark 5.3.8 (Recognizing Extensions of δ_n). Let $s \in [n]$ and define $A = [n] - \{s\}$. Suppose

$$p = (\delta_1, \ldots, \delta_{i-1}, \delta_{e_s}, \delta_{i+1}, \ldots, \delta_k)$$

is a partial factorization of δ_n. Then via a sequence of Hurwitz moves and the remark above, we may see that p is an extension of the partial factorization

$$(\delta_1, \ldots, \delta_{i-1}, \delta_{i+1}^{\delta_{e_s}}, \ldots, \delta_k^{\delta_{e_s}})$$

for δ_A. In other words, if a partial factorization for δ_n contains δ_{e_s}, then it may be obtained from a partial factorization for A via a combination of Hurwitz moves and the extensions described above.

The situation is similar for the case when $B = [n]$ and $A \subseteq B$ is arbitrary; the only difference is that we do not require that the inserted dual simple braid is a positive half-twist.

Remark 5.3.9 (Extending Factorizations of δ_A). Let $A \subseteq B \subseteq [n]$. Similar to the case for Remark 5.3.7, each partial factorization of δ_A may be extended into a partial factorization of δ_B by inserting ${\delta_A}^{-1}\delta_B$ as the new first entry and shifting the others down. We may then similarly consider the Hurwitz moves which move the dual simple braid ${\delta_A}^{-1}\delta_B$ into each possible position. A partial factorization of δ_B is then recognized as an extension of a partial factorization for δ_A if and only if it contains the dual simple braid ${\delta_A}^{-1}\delta_B$.

Example 5.3.10. Define the subsets $A = \{1, 2, 5, 6\}$ and $B = \{1, 2, 3, 5, 6\}$ of $[6]$ and consider the partial factorization $p = (2\ 5)(1\ 2\ 6)$ of δ_A, where we denote $\delta_{\{2,5\}}$ and $\delta_{\{1,2,6\}}$ by their associated permutations and, for the sake of brevity, we omit the parentheses for p and simply write the ordered product of its entries. Then there are three extensions of p into a partial factorization for δ_B:

$$(2\ 5)(1\ 2\ 6)(3\ 5)$$

$$(2\ 5)(3\ 5)(1\ 2\ 6)$$

$$(3\ 5)(2\ 3)(1\ 2\ 6)$$

(2 5)(1 2 6)(3 5)(4 5)	(2 5) (1 2 6)(4 5)(3 4)	(2 5) (4 5)(1 2 6)(3 4)	(4 5) (2 4)(1 2 6)(3 4)
(2 5) (3 5)(1 2 6)(4 5)	(2 5) (3 5)(4 5)(1 2 6)	(2 5) (4 5)(3 4)(1 2 6)	(4 5) (2 4)(3 4)(1 2 6)
(3 5) (2 3)(1 2 6)(4 5)	(3 5)(2 3) (4 5)(1 2 6)	(3 5)(4 5) (2 3)(1 2 6)	(4 5)(3 4) (2 3)(1 2 6)

Figure 5.11: The 12 factorizations of δ_6 obtained from the partial factorization $(2\ 5)(1\ 2\ 6) = (1\ 2\ 5\ 6)$

Then each of these partial factorizations can be extended to one of δ_6 in four ways, yielding a total of 12 partial factorizations which arise from p, displayed in Figure 5.11. Partial factorizations produced in this way exhibit a structure which may be seen via a certain topological space - see Section 6.5.

5.4 Boundary Braids and Permutations

Considering dual simple braids as motions in a configuration space allows us to classify elements of NC_n based on their action on the vertices of D_n. In this section, we introduce several subposets of NC_n, each determined by requiring a certain type of action on a subset of the vertex set P_n.

Definition 5.4.1 (Boundary permutations)**.** Let $B \subseteq [n]$. For all $\pi \in \mathrm{NC}_n$, the noncrossing permutation $\sigma_\pi \in \mathrm{NP}_n$ is a $(B, \cdot)$-*boundary permutation* if for all $b \in B$, $b \cdot \sigma_\pi \in \{b, b+1\}$, reduced mod n. Similarly, we say that σ_π is a $(\cdot, B)$-*boundary permutation* if $\sigma_\pi \cdot b \in \{b, b-1\}$, reduced mod n. Refer to the sets of all $(B, \cdot)$-boundary and $(\cdot, B)$-boundary permutations as $\mathrm{NP}_n(B, \cdot)$ and $\mathrm{NP}_n(\cdot, B)$, respectively. Notice that if $\sigma_\pi \in \mathrm{NP}_n(B, \cdot)$ and $B \cdot \sigma_\pi = C$, then $B = \sigma_\pi \cdot C$ and thus $\sigma_\pi \in \mathrm{NP}_n(\cdot, C)$.

Definition 5.4.2 (Boundary braids)**.** Let $B \subseteq [n]$. For all $\pi \in \mathrm{NC}_n$, the dual simple braid $\delta_\pi \in \mathrm{DS}_n$ is a $(B,\cdot)$-*boundary braid* if there is a representative f for δ_π such that, for all $b \in B$, the $(b,\cdot)$-strand f^b terminates at either b or $b+1$. Similarly, δ_π is a $(\cdot,B)$-*boundary braid* if there is a representative f for δ_π such that the $(\cdot,b)$-strand f_b begins at either b or $b-1$, for all $b \in B$. We refer to the sets of dual simple braids which are $(B,\cdot)$-boundary and $(\cdot,B)$-boundary braids as $\mathrm{DS}_n(B,\cdot)$ and $\mathrm{DS}_n(\cdot,B)$, respectively. Notice that each $(B,\cdot)$-boundary braid in DS_n has a representative where the f^b strand remains in the boundary of D_n for each $b \in B$. Generalizing this notion to all of $\textsc{Braid}_n$ is the topic of Chapter 7.

Definition 5.4.3 (Boundary partitions)**.** Let $B \subseteq [n]$. If $\pi \in \mathrm{NC}_n$, we say that π is a $(B,\cdot)$-*boundary partition* if each $b \in B$ either shares a block with $b+1$ (reduced mod n) or forms a singleton block $\{b\} \in \pi$. Similarly, π is a $(\cdot,B)$-*boundary partition* if each $b \in B$ either shares a block with $b-1$ (reduced mod n) or forms a singleton block $\{b\} \in \pi$. We refer to the sets of all $(B,\cdot)$-boundary and $(\cdot,B)$-boundary partitions as $\mathrm{NC}_n(B,\cdot)$ and $\mathrm{NC}_n(\cdot,B)$, respectively.

Example 5.4.4. Let $B = \{2,4,5\} \subseteq [5]$. Then $\mathrm{NC}_5(B,\cdot)$ is a subposet of NC_5 with 12 elements, depicted in Figure 5.12.

Proposition 5.4.5 (Boundary subposets are isomorphic)**.** *Let $B \subseteq [n]$. Then the natural identifications between NC_n, DS_n, and NP_n restrict to the isomorphisms*

$$\mathrm{NC}_n(B,\cdot) \cong \mathrm{DS}_n(B,\cdot) \cong \mathrm{NP}_n(B,\cdot)$$

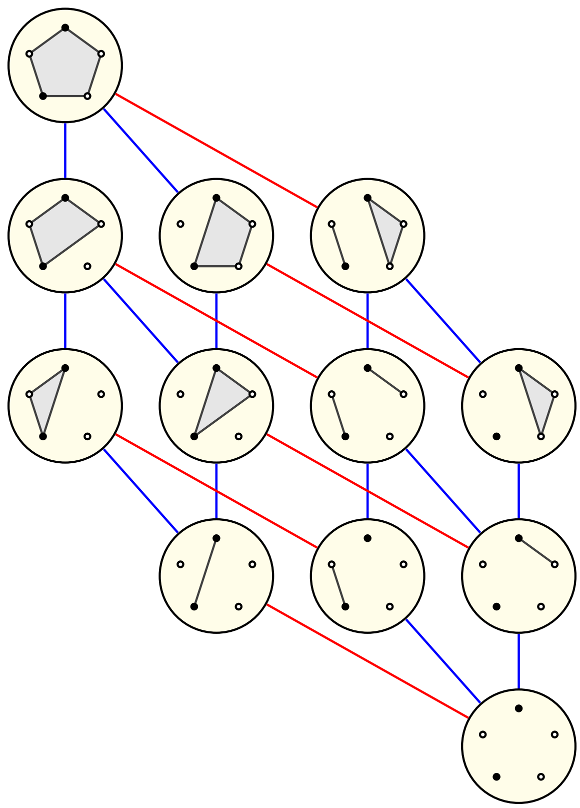

Figure 5.12: The poset of boundary partitions $\mathrm{NC}_5(\{2,4,5\},\cdot)$, where the top vertex of each noncrossing partition is labeled by 1 and elements of B are labeled by a white dot. The blue and red edge labels serve to illustrate the direct product structure described in Theorem 5.4.16.

and

$$\mathrm{NC}_n(\cdot, B) \cong \mathrm{DS}_n(\cdot, B) \cong \mathrm{NP}_n(\cdot, B).$$

Moreover, there is an anti-isomorphism $\mathrm{NP}_n(B, \cdot) \to \mathrm{NP}_n(\cdot, B)$ *which provide analogous maps for* NC_n *and* DS_n*.*

Proof. Fix $B \subseteq [n]$. Let $\pi \in \mathrm{NC}_n(B, \cdot)$ and consider the corresponding dual simple braid $\delta_\pi \in \mathrm{DS}_n$. For each $b \in B$, if $\{b\}$ is a block in π, then δ_π has a representative f which fixes the $(b, \cdot)$ strand. Similarly, if b and $b+1$ share a block, then every representative of δ_π has a $(b, \cdot)$ strand terminates at $b+1$. Hence, $\delta_\pi \in \mathrm{DS}_n(B, \cdot)$. Similarly, the noncrossing permutation σ_π either has $b \cdot \sigma_\pi = b$ or $b \cdot \sigma_\pi = b + 1$ for each $b \in B$, so $\sigma_\pi \in \mathrm{NP}_n(B, \cdot)$. Hence, the maps sending $\pi \mapsto \delta_\pi$ and $\pi \mapsto \sigma_\pi$ restrict to the prescribed isomorphisms and one can easily see that the same argument holds in the $(\cdot, B)$ case.

Now, suppose $\sigma_\pi \in \mathrm{NP}_n(B, \cdot)$. Then $b \cdot \sigma_\pi \in \{b, b+1\}$, so we either have $\sigma_\pi \cdot b = b$ or $\sigma_\pi \cdot (b+1) = b$. By Theorem 5.2.8 (applied to noncrossing permutations), we know that ${\sigma_\pi}^{-1}\sigma_n \in \mathrm{NP}_n$ and thus by Proposition 5.2.9, $\sigma_n {\sigma_\pi}^{-1} \in \mathrm{NP}_n$ as well. If $\sigma_\pi \cdot b = b$, then

$$(\sigma_n {\sigma_\pi}^{-1}) \cdot b = \sigma_n \cdot b = b - 1.$$

If $\sigma_\pi \cdot (b+1) = b$, then

$$(\sigma_n {\sigma_\pi}^{-1}) \cdot b = \sigma_n \cdot (b+1) = b.$$

Hence, $(\sigma_n {\sigma_\pi}^{-1}) \cdot b \in \{b, b-1\}$ and thus $\sigma_n {\sigma_\pi}^{-1} \in \mathrm{NP}_n(\cdot, B)$.

The map $\mathrm{NP}_n(B, \cdot) \to \mathrm{NP}_n(\cdot, B)$ which sends $\sigma_\pi \mapsto \sigma_n {\sigma_\pi}^{-1}$ is an anti-isomorphism. To see this, it suffices to check the covering relations. Suppose $\sigma_{\pi_1} < \sigma_{\pi_2}$ in $\mathrm{NP}_n(B, \cdot)$

and that ${\sigma_{\pi_1}}^{-1}\sigma_{\pi_2} = \sigma_e$ is a transposition. Then

$$\begin{aligned}\sigma_n{\sigma_{\pi_2}}^{-1} &= \sigma_n(\sigma_{\pi_1}\sigma_e)^{-1}\\ &= \sigma_n{\sigma_e}^{-1}{\sigma_{\pi_1}}^{-1}\\ &= (\sigma_n{\sigma_{\pi_1}}^{-1})(\sigma_{\pi_1}{\sigma_e}^{-1}{\sigma_{\pi_1}}^{-1})\end{aligned}$$

and thus

$$(\sigma_n{\sigma_{\pi_2}}^{-1})(\sigma_{\pi_1}\sigma_e{\sigma_{\pi_1}}^{-1}) = \sigma_n{\sigma_{\pi_1}}^{-1}.$$

Since the set of transpositions in $\textsc{Sym}_n$ is closed under conjugation, we know that $\sigma_{\pi_1}\sigma_e{\sigma_{\pi_1}}^{-1}$ is a transposition and thus $\sigma_n{\sigma_{\pi_2}}^{-1}$ is covered by $\sigma_n{\sigma_{\pi_1}}^{-1}$ in $\mathrm{NP}_n(\cdot, B)$. This argument is easily reversed, and hence we have an order-reversing bijection between covering relations in $\mathrm{NP}_n(B, \cdot)$ and $\mathrm{NP}_n(\cdot, B)$. □

Remark 5.4.6. For the remainder of this chapter, we focus our efforts on $\mathrm{NC}_n(B, \cdot)$ and the corresponding subposets of DS_n and NP_n, noting that by Proposition 5.4.5, analogous results hold for $\mathrm{NC}_n(\cdot, B)$. As we see when applying the tools from this section in Chapter 7, it suffices to consider this case.

We can characterize the posets defined above by properties satisfied by their maximal chains.

Proposition 5.4.7. *Let $B \subseteq [n]$ and for each $b \in B$, define δ_{e_b} to be the positive half-twist which swaps p_b and p_{b+1}. Then the maximal chains of $\mathrm{NC}_n(B, \cdot)$ are precisely the maximal chains of NC_n which correspond to factorizations of δ_n which include δ_{e_b} for all $b \in B$.*

Proof. Suppose $(\delta_{\pi_1}, \dots, \delta_{\pi_{n-1}})$ is a factorization of δ_n into positive half-twists with the property that, for some $b \in B$, $\delta_{\pi_i} \neq \delta_{e_b}$ for all i. Then we can fix the smallest $j \in [n-1]$ such that $b \cdot \delta_{\pi_j} \neq b$; one must exist since $b \cdot \delta_n \neq b$. By assumption, $b \cdot \delta_{\pi_i} = b$ for all $1 \leq i < j$, so we then have that

$$b \cdot (\delta_{\pi_1}\delta_{\pi_2} \cdots \delta_{\pi_j}) \notin \{b, b+1\},$$

but since this labels an element along our maximal chain in $\mathrm{NC}_n(B, \cdot)$, we have a contradiction. $\square$

Remark 5.4.8. In the language established by Remarks 5.3.7 and 5.3.8, the factorizations of δ_n corresponding to maximal chains in $\mathrm{NC}_n(B, \cdot)$ are those obtained by extending factorizations of δ_A to factorizations of δ_n, where $A = [n] - B$.

Remark 5.4.9. Notice that when $\pi_1 \leq \pi_2$ in $\mathrm{NC}_n(B, \cdot)$, we must have ${\delta_{\pi_1}}^{-1}\delta_{\pi_2} \in \mathrm{DS}_n$, but this element is not generally in $\mathrm{DS}_n(B, \cdot)$. Example 5.4.4 contains an example of this: $\delta_{\{1,2,3,4\}} \leq \delta_5$ in $\mathrm{NC}_5(\{2,4,5\}, \cdot)$, but ${\delta_{\{1,2,3,5\}}}^{-1}\delta_5 = \delta_{\{4,5\}}$, which is not an element of $\mathrm{DS}_5(\{2,4,5\}, \cdot)$.

We spend the remainder of the section exhibiting a decomposition of $\mathrm{NC}_n(B, \cdot)$ as the direct product of two subposets. To this end, we define two maps of $\mathrm{NC}_n(B, \cdot)$ to itself which lead to useful subposets.

Definition 5.4.10 ($\textsc{Fix}_B$)**.** Fix $B \subseteq [n]$. Define the map $\textsc{Fix}_B : \mathrm{NC}_n(B, \cdot) \to \mathrm{NC}_n(B, \cdot)$ by sending each $\pi \in \mathrm{NC}_n(B, \cdot)$ to the noncrossing partition $\textsc{Fix}_B(\pi)$ obtained by removing each element $b \in B$ from its original block in π and including the singleton block

$\{b\}$. The effect on the corresponding noncrossing permutation σ_π is that each element of B is removed from the cycle in which it appears. Since we obtain $\textsc{Fix}_B(\pi)$ from π by breaking apart blocks in π, notice that $\textsc{Fix}_B(\pi) \leq \pi$ in the refinement order for NC_n. Moreover, if $\pi_1 \leq \pi_2$, then each block of π_1 is contained in a block of π_2; by removing the elements of B in each, we also have $\textsc{Fix}_B(\pi_1) \leq \textsc{Fix}_B(\pi_2)$. In other words, $\textsc{Fix}_B$ is an order-preserving map. We similarly define $\textsc{Fix}_B$ on $\mathrm{DS}_n(B, \cdot)$ and $\mathrm{NP}_n(B, \cdot)$ and make clear from context which we consider to be the domain.

Definition 5.4.11 ($\textsc{Move}_B$)**.** Fix $B \subseteq [n]$ and let $\pi \in \mathrm{NC}_n(B, \cdot)$. By Theorem 5.2.8, $\textsc{Fix}_B(\delta_\pi)^{-1}\delta_\pi$ is a dual simple braid, and this is an element of $\mathrm{DS}_n(B, \cdot)$ since the corresponding noncrossing permutation satisfies $b \cdot \textsc{Fix}_B(\sigma_\pi)^{-1}\sigma_\pi = b \cdot \sigma_\pi \in \{b, b+1\}$ and thus $\textsc{Fix}_B(\sigma_\pi)^{-1}\sigma_\pi \in \mathrm{NP}_n(B, \cdot)$. Define the noncrossing partition corresponding to this element to be $\textsc{Move}_B(\pi)$. Then the associated dual simple braids satisfy $\textsc{Fix}_B(\delta_\pi)\textsc{Move}_B(\delta_\pi) = \delta_\pi$ for all $\pi \in \mathrm{NC}_n(B, \cdot)$. As in Definition 5.4.10, we denote by $\textsc{Move}_B$ compatible maps on each of $\mathrm{NC}_n(B, \cdot)$, $\mathrm{DS}_n(B, \cdot)$, and $\mathrm{NP}_n(B, \cdot)$.

Proposition 5.4.12. *Let $B \subseteq [n]$. Then the map $\textsc{Move}_B : \mathrm{NC}_n(B, \cdot) \to \mathrm{NC}_n(B, \cdot)$ is order-preserving.*

Proof. By Proposition 5.4.5, it suffices to prove this for the analogously defined map on $\mathrm{NP}_n(B, \cdot)$.

Suppose $\sigma_{\pi_1} < \sigma_{\pi_2}$ is a covering relation in $\mathrm{NP}_n(B, \cdot)$. Then $\sigma_{\pi_2} = \sigma_{\pi_1}\sigma_e$ for some transposition σ_e, where $e = \{i, j\} \subseteq [n]$. If σ_{π_1} is a (B, B')-boundary permutation, then $\sigma_e \in \mathrm{NP}_n(B', \cdot)$. If either i or j is an element of B', then $\textsc{Fix}_{B'}(\sigma_e)$ is the identity permutation. If neither i nor j is an element of B, then $\textsc{Fix}_{B'}(\sigma_e) = \sigma_e$. In either case, we have $\textsc{Fix}_B(\sigma_{\pi_2}) = \textsc{Fix}_B(\sigma_{\pi_1})\textsc{Fix}_{B'}(\sigma_e)$.

Applying the definition of $\textsc{Move}_B$, we have the following sequence of equalities:

$$\begin{aligned}
\textsc{Move}_B(\sigma_{\pi_2}) &= \textsc{Fix}_B(\sigma_{\pi_2})^{-1}\sigma_{\pi_2} \\
&= \textsc{Fix}_B(\sigma_{\pi_1}\sigma_e)^{-1}\sigma_{\pi_1}\sigma_e \\
&= \textsc{Fix}_{B'}(\sigma_e)^{-1}\textsc{Fix}_B(\sigma_{\pi_1})^{-1}\sigma_{\pi_1}\sigma_e \\
&= \textsc{Fix}_{B'}(\sigma_e)^{-1}\textsc{Move}_B(\sigma_{\pi_1})\sigma_e \\
&= \textsc{Move}_B(\sigma_{\pi_1})\textsc{Move}_B(\sigma_{\pi_1})^{-1}\textsc{Fix}_{B'}(\sigma_e)^{-1}\textsc{Move}_B(\sigma_{\pi_1})\sigma_e \\
&= \textsc{Move}_B(\sigma_{\pi_1})({\sigma_{\pi_1}}^{-1}\textsc{Fix}_B(\sigma_{\pi_1}))\textsc{Fix}_{B'}(\sigma_e)^{-1}(\textsc{Fix}_B(\sigma_{\pi_1})^{-1}\sigma_{\pi_1}) \\
&= \textsc{Move}_B(\sigma_{\pi_1}){\sigma_{\pi_1}}^{-1}\textsc{Fix}_B(\sigma_{\pi_1})\textsc{Fix}_B(\sigma_{\pi_2})^{-1}\sigma_{\pi_1} \\
&= \textsc{Move}_B(\sigma_{\pi_1}){\sigma_{\pi_1}}^{-1}\textsc{Fix}_{B'}(\sigma_e)^{-1}\sigma_{\pi_1}
\end{aligned}$$

If either i or j is an element of B', then $\textsc{Fix}_{B'}(\sigma_e)$ is the identity permutation and the righthand side is equal to $\textsc{Move}_B(\sigma_{\pi_1})\sigma_e$. On the other hand, suppose neither i nor j is in B'. Then $\textsc{Fix}_{B'}(\sigma_e) = \sigma_e$ and thus we have

$$\textsc{Move}_B(\sigma_{\pi_2}) = \textsc{Move}_B(\sigma_{\pi_1})({\sigma_{\pi_1}}^{-1}\sigma_e\sigma_{\pi_1}).$$

Since $\sigma_{\pi_2} = \sigma_{\pi_1}\sigma_e$, we know that the pair $(\sigma_{\pi_1}, \sigma_e)$ is properly ordered, so by Proposition 5.3.5, the righthand side is the product of $\textsc{Move}_B(\sigma_{\pi_1})$ and a noncrossing permutation. In either case, Theorem 5.2.8 tells us that $\textsc{Move}_B(\sigma_{\pi_1}) \leq \textsc{Move}_B(\sigma_{\pi_2})$ and we're done. $\square$

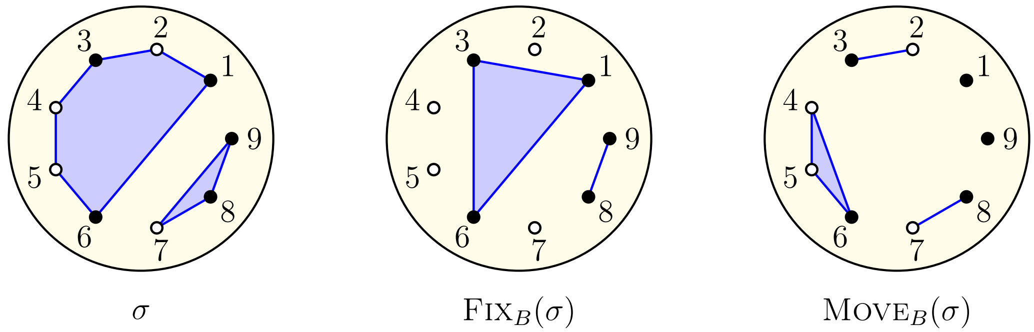

Figure 5.13: The noncrossing partitions corresponding to σ, $\textsc{Fix}_B(\sigma)$, and $\textsc{Move}_B(\sigma)$ as described in Example 5.4.13. In this example, $B = \{2, 4, 5, 7\}$ and each corresponding vertex in P_B is denoted by a white dot.

Example 5.4.13. Let $\sigma = (1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9)$ and $B = \{2, 4, 5, 7\}$. Then we have $\textsc{Fix}_B(\sigma) = (1\ 3\ 6)(8\ 9)$ and $\textsc{Move}_B(\sigma) = (2\ 3)(4\ 5\ 6)(7\ 8)$. See Figure 5.13.

Remark 5.4.14 (Minima and maxima). Let $B \subseteq [n]$ and consider the two subposets $\textsc{Fix}_B(\mathrm{DS}_n(B, \cdot))$ and $\textsc{Move}_B(\mathrm{DS}_n(B\cdot))$ of $\mathrm{DS}_n(B, \cdot)$. Since $\textsc{Fix}_B$ and $\textsc{Move}_B$ are order-preserving maps, their images are bounded with minimum element $\textsc{Fix}_B(\hat{0}) = \hat{0}$ and $\textsc{Move}_B(\hat{0}) = \textsc{Fix}_B(\hat{0})^{-1}\hat{0} = \hat{0}$. We may similarly find the maximal elements by applying $\textsc{Fix}_B$ and $\textsc{Move}_B$ to the maximal element $\delta_n \in \mathrm{DS}_n(B, \cdot)$. If $A = [n] - B$, then $\textsc{Fix}_B(\delta_n) = \delta_A$ is the maximum element of $\textsc{Fix}_B(\mathrm{DS}_n(B, \cdot))$. Similarly, the maximum element of $\textsc{Move}_B(\mathrm{DS}_n(B, \cdot))$ is $\textsc{Move}_B(\delta_n) = {\delta_A}^{-1}\delta_n$.

Proposition 5.4.15. *Let $B \subseteq [n]$. Then the intersection of $\textsc{Fix}_B(\mathrm{NC}_n(B, \cdot))$ and $\textsc{Move}_B(\mathrm{NC}_n(B, \cdot))$ contains only the minimum element $\hat{0}$.*

Proof. This follows directly from the definitions. If $\pi \in \textsc{Fix}_B(\mathrm{NC}_n(B,\cdot))$, then $\{b\}$ is a block in π for each $b \in B$. In other words, the dual simple braid δ_π has a representative in which each $(b,\cdot)$-strand is fixed.

If $\pi \in \textsc{Move}_B(\mathrm{NC}_n(B,\cdot))$, then the dual simple braid δ_π is equal to $\textsc{Fix}_B(\delta_{\pi'})^{-1}\delta_{\pi'}$ for some $\pi' \in \mathrm{NC}_n(B,\cdot)$. We then have that $\textsc{Fix}_B(\delta_{\pi'})\delta_\pi = \delta_{\pi'}$, and since we assumed that $\delta_\pi \in \textsc{Fix}_B(\mathrm{DS}_n(B,\cdot))$, we know that $\delta_{\pi'}$ has a representative in which the $(b,\cdot)$-strand is fixed for each $b \in B$. However, we then know that $\textsc{Fix}_B(\delta_{\pi'}) = \delta_{\pi'}$ and thus

$$\delta_\pi = \textsc{Fix}_B(\delta_{\pi'})^{-1}\delta_{\pi'} = {\delta_{\pi'}}^{-1}\delta_{\pi'}$$

and therefore δ_π is the identity braid. Hence, $\pi = \hat{0}$. □

The maps $\textsc{Fix}_B$ and $\textsc{Move}_B$ provide more than a useful identity: they are the projection maps for a direct product structure on the poset of B-boundary permutations.

Theorem 5.4.16. *Let $B \subseteq [n]$. Then $\mathrm{NC}_n(B,\cdot)$ is isomorphic to the direct product of the subposets $\textsc{Fix}_B(\mathrm{NC}_n(B,\cdot))$ and $\textsc{Move}_B(\mathrm{NC}_n(B,\cdot))$.*

Proof. Since $\textsc{Fix}_B$ and $\textsc{Move}_B$ are order-preserving maps on $\mathrm{NC}_n(B,\cdot)$, the map which sends σ to $(\textsc{Fix}_B(\sigma), \textsc{Move}_B(\sigma)) \in \textsc{Fix}_B(\mathrm{NC}_n(B,\cdot)) \times \textsc{Move}_B(\mathrm{NC}_n(B,\cdot))$ is order-preserving as well. To see that this map is a bijection, suppose that $\textsc{Fix}_B(\sigma_1) = \textsc{Fix}_B(\sigma_2)$ and $\textsc{Move}_B(\sigma_1) = \textsc{Move}_B(\sigma_2)$. Then by definition of $\textsc{Move}_B$ we have

$$\textsc{Fix}_B(\sigma_1)^{-1}\sigma_1 = \textsc{Fix}_B(\sigma_2)^{-1}\sigma_2$$

and thus $\sigma_1 = \sigma_2$.

Moreover, the inverse of this map is order-preserving. To see this, we show that each covering relation in the direct product corresponds to a covering relation in $\mathrm{NC}_n(B,\cdot)$. Recall that covering relations in a poset $P \times Q$ are those of the form $(p,q) < (p',q')$ where either $p = p'$ and $q < q'$ is a covering relation in Q, or vice versa. We will treat each of these cases separately.

Let $\sigma_1, \sigma_2 \in \mathrm{NP}(B,\cdot)$. Suppose $\textsc{Fix}_B(\sigma_1) = \textsc{Fix}_B(\sigma_2)$ and $\textsc{Move}_B(\sigma_1) < \textsc{Move}_B(\sigma_2)$ is a covering relation in $\textsc{Move}_B(\mathrm{NP}_n(B,\cdot))$. Then there is a transposition σ_e such that $\textsc{Move}_B(\sigma_1)\sigma_e = \textsc{Move}_B(\sigma_2)$. Combining these equalities, we have

$$\textsc{Fix}_B(\sigma_1)\textsc{Move}_B(\sigma_1)\sigma_e = \textsc{Fix}_B(\sigma_2)\textsc{Move}_B(\sigma_2)$$

and by Definition 5.4.11, $\sigma_1\sigma_e = \sigma_2$.

Now, suppose $\textsc{Fix}_B(\sigma_1) < \textsc{Fix}_B(\sigma_2)$ is a covering relation in $\textsc{Fix}_B(\mathrm{NP}_n(B,\cdot))$ and that $\textsc{Move}_B(\sigma_1) = \textsc{Move}_B(\sigma_2)$. Then $\textsc{Fix}_B(\sigma_1)\sigma_e = \textsc{Fix}_B(\sigma_2)$ for some transposition σ_e and thus

$$\textsc{Fix}_B(\sigma_1)\sigma_e\textsc{Move}_B(\sigma_1) = \textsc{Fix}_B(\sigma_2)\textsc{Move}_B(\sigma_2).$$

The lefthand side may be rewritten as $\textsc{Fix}_B(\sigma_1)\textsc{Move}_B(\sigma_1){\sigma_e}^{\textsc{Move}_B(\sigma_1)}$ and is thus equal to $\sigma_1{\sigma_e}^{\textsc{Move}_B(\sigma_1)}$. Moreover, the righthand side is equal to σ_2. Since the conjugate of a transposition is again a transposition, we thus have $\sigma_1 < \sigma_2$.

By the isomorphisms given in Proposition 5.4.5, the work above may be translated into $\mathrm{NC}_n(B,\cdot)$ and thus the map which sends $\mathrm{NC}_n(B,\cdot)$ to the direct product $\textsc{Fix}_B(\mathrm{NC}_n(B,\cdot))$ and $\textsc{Move}_B(\mathrm{NC}_n(B,\cdot))$ is an isomorphism and we are done. $\square$

5.5 The Dual Presentation

Since the dual simple braids contain the usual $n-1$ half-twists which generate the n-strand braid group, they certainly form a larger generating set for $\textsc{Braid}_n$ - albeit with a more complicated presentation. This presentation for the braid group, articulated in the early 2000s [Bra01] [Bes03], is an essential tool in our perspective for the braid group.

Instead of generating $\textsc{Braid}_n$ by the usual set of $n-1$ "adjacent" half-twists, we are interested in the larger generating set of all nontrivial dual simple braids. We then impose every relation of the form $\beta_1\beta_2 = \beta$, where β_1, β_2, and β are nontrivial dual simple braids. Fitting the theme of this chapter, this group-theoretic condition has a combinatorial description: proper ordering. Referring back to Proposition 5.3.4, we are able to state the following theorem [Bra01] [Bes03].

Theorem 5.5.1 (Dual Presentation). *Let* DS_n^* *denote the set of nontrivial dual simple braids. Then* DS_n^* *generates the braid group with the following presentation:*

$$\textsc{Braid}_n = \langle \mathrm{DS}_n^* \mid \delta_{\pi_1}\delta_{\pi_2} = \delta_{\pi_1 \vee \pi_2} \textit{ if } (\pi_1, \pi_2) \textit{ properly ordered } \rangle$$

This is referred to as the dual presentation *for the braid group.*

Despite the use of the term "dual", the name for this presentation does not arise from an operation on group presentations which produces this one from the usual presentation. Instead, the name "dual presentation" refers to the presence of certain numerological dualities between the two. For more information on the two presentations, see [Arm09].

Among the benefits obtained via the dual presentation is the ability to translate the combinatorics of noncrossing partitions into useful tools for the braid group. For example, the subposets of NC_n defined in the preceding chapter lead to important subsets of the braid group.

5.6 Dual Parabolics and Fixed Subgroups

Recall that $S = \{\sigma_1, \ldots, \sigma_{n-1}\}$ with $\sigma_i = (i\ i+1)$ is the standard generating set for the symmetric group SYM_n, and any subset $I \subseteq S$ of the generators produces a *parabolic subgroup* W_I of the symmetric group, each of which is isomorphic to a direct product of smaller symmetric groups. The same is true for the standard presentation for the braid group. While these subgroups are useful in the general study of Coxeter and Artin groups, there is a natural generalization in the context of the dual presentation for BRAID_n.

Definition 5.6.1 (Irreducible Dual Parabolics in BRAID_n)**.** Let $A \subseteq [n]$. Then the subgroup $\mathrm{BRAID}_A = \langle \delta_{A'} \mid A' \subseteq A \rangle$ is the *irreducible dual parabolic subgroup* of BRAID_n associated to A. We say that BRAID_A is *maximal* if $A = [n] - \{s\}$ for some $s \in [n]$.

Irreducible dual parabolics are a familiar class of groups, but to prove this we require the following lemma. Recall from Remark 4.2.5 that we have a canonical choice of basepoint when considering BRAID_n as a fundamental group of a configuration space.

Lemma 5.6.2 (Isomorphic groups)**.** *For each subset $A \subseteq [n]$ of size k, let $B = [n] - A$ and $D^B = D_n - P_B$. The natural inclusion map $D_A \hookrightarrow D^B$ extends to an inclusion* *map* $h : \text{UC}\textsc{onf}_k(D_A) \hookrightarrow \text{UC}\textsc{onf}_k(D^B)$ *and the induced map on fundamental groups* $h_*\colon \pi_1(\text{UC}\textsc{onf}_k(D_A), P_A) \to \pi_1(\text{UC}\textsc{onf}_k(D^B), P_A)$ *is an isomorphism.*

Proof. When $k = 1$ both groups are trivial and there is nothing to prove. For each element $[f]$ in $\pi_1(\text{UC}\textsc{onf}_k(D^B), P_A)$, the path f can be homotoped so that it never leaves the subdisk D_A. One can, for example, modify f so that the configurations first radially shrink towards a point in the interior of D_A, followed by the original representative f on a rescaled version of D^B strictly contained in D_A, followed by a radial expansion back to the starting position. This shows that h_* is onto. Suppose $[f]$ and $[g]$ are elements in $\pi_1(\text{UC}\textsc{onf}_k(D_A), P_A)$ such that f and g are homotopic based paths in the bigger space $\text{UC}\textsc{onf}_k(D^B)$. A very similar modification that can be done here so that the entire homotopy between f and g takes place inside the subdisk D_A, and this shows that h_* is injective. $\square$

Proposition 5.6.3 ($\textsc{Fix}_n(B) = \textsc{Braid}_A$)**.** *Let* $A \subseteq [n]$ *and define* $B = [n] - A$. *Then the fixed subgroup* $\textsc{Fix}_n(B)$ *is equal to the dual parabolic subgroup* $\textsc{Braid}_A$.

Proof. Since each rotation braid $\delta_{A'}$ with $A' \subseteq A$ fixes P_B, it is clear that $\textsc{Braid}_A$ is a subset of $\textsc{Fix}_n(B)$. Now, let $\beta \in \textsc{Fix}_n(B)$. Then β has a representative which fixes each $(b, \cdot)$-strand. Since strands have non-intersecting drawings, each $(a, \cdot)$-strand with $a \in A$ is then a path $f^a : [0, 1] \to D_n - P_B$. That is, each $(a, \cdot)$-strand represents an element of $\pi_1(\text{UC}\textsc{onf}_{|A|}(D^B), P_A)$ and thus by Lemma 5.6.2 an element of $\pi_1(\text{UC}\textsc{onf}_{|A|}(D_A), P_A)$. Hence, $\textsc{Fix}_n(B)$ is a subgroup of $\textsc{Braid}_A$ and thus $\textsc{Fix}_n(B) = \textsc{Braid}_A$. $\square$

Definition 5.6.4 (Dual Parabolics)**.** Let $\pi \in \mathrm{NC}_n$. Then the subgroup

$$\textsc{Braid}_\pi = \langle \delta_{\pi'} \mid \pi' \leq \pi \rangle$$

is the *dual parabolic subgroup* of $\textsc{Braid}_n$ associated to π. Note that if $A_1, \ldots, A_k$ are the blocks of π, then

$$\textsc{Braid}_\pi \cong \textsc{Braid}_{A_1} \times \cdots \times \textsc{Braid}_{A_k}$$

and thus every dual parabolic subgroup is the direct product of irreducible dual parabolics.

Similar to the case for standard parabolic subgroups, the isomorphism type of dual parabolics and their intersections are easily computed and form useful tools for the dual presentation of the braid group.

Proposition 5.6.5. *Let $A \subseteq [n]$ with $|A| = k$. Then $\textsc{Braid}_A$ is a subgroup of $\textsc{Braid}_n$ which is isomorphic to $\textsc{Braid}_k$.*

Proof. By Lemma 5.6.2 and Proposition 5.6.3, we may view $\textsc{Braid}_A$ as the fundamental group $\pi_1(\mathrm{U}\textsc{Conf}_k(D_A), P_A)$. Since D_A is naturally homeomorphic to the standard k-gon D_k and this map sends P_A to P_k, by Proposition 4.2.4 we have the isomorphism

$$\pi_1(\mathrm{U}\textsc{Conf}_k(D_A), P_A) \cong \pi_1(\mathrm{U}\textsc{Conf}_k(D_k), P_k)$$

and thus $\textsc{Braid}_A \cong \textsc{Braid}_k$. $\square$

In particular, every dual parabolic is the direct product of braid groups of smaller rank. We may now prove that the intersections of irreducible dual parabolics are smaller irreducible dual parabolics.

Lemma 5.6.6 (Maximal dual parabolics)**.** *The intersection of two maximal irreducible dual parabolic subgroups is an irreducible dual parabolic subgroup. In particular, for all* $n > 0$ *and for all* $i, j \in [n]$,

$$\textsc{Fix}_n(\{i,j\}) = \textsc{Fix}_n(\{i\}) \cap \textsc{Fix}_n(\{j\}).$$

Proof. For every pair of vertices p_i and p_j one can select a sequence $E = (e_1, \ldots, e_{n-1})$ of edges in D_n so that together, in this order, they form an embedded path through all of the vertices of D_n, starting at p_i and ending at p_j. The corresponding positive half-twists atisfy the usual relations that δ_{e_i} and δ_{e_j} commute when e_i and e_j are disjoint, and $\delta_{e_i}\delta_{e_2j}\delta_{e_i} = \delta_{e_j}\delta_{e_i}\delta_{e_j}$ when e_i and e_j share a single vertex. Thus, $\{e_1, \ldots, e_{n-1}\}$ forms a generating set for $\textsc{Braid}_n$, and the subsets $\{e_1, \ldots, e_{n-2}\}$ and $\{e_2, \ldots, e_{n-1}\}$ generate the subgroups $\textsc{Fix}_n(\{j\})$ and $\textsc{Fix}_n(\{i\})$, respectively. Similarly, $\{e_2, \ldots, e_{n-2}\}$ generates the subgroup $\textsc{Fix}_n(\{i,j\})$. By Proposition 2.3.3, we are done. $\square$

Lemma 5.6.7 (Relative maximal dual parabolics)**.** *The intersection of two irreducible dual parabolic subgroups that are both maximal in a third irreducible dual parabolic subgroup is again an irreducible dual parabolic subgroup. In other words, for all* $n > 0$, $i, j \in [n]$, *and* $C \subseteq [n]$, *we have*

$$\textsc{Fix}_n(C \cup \{i,j\}) = \textsc{Fix}_n(C \cup \{i\}) \cap \textsc{Fix}_n(C \cup \{j\})$$

Proof. When C is empty, the statement is just Lemma 5.6.6 and when C is $[n]$ there is nothing to prove. When C is proper and non-empty, all three groups are contained in $\textsc{Fix}_n(C) = \textsc{Braid}_A \cong \textsc{Braid}_k$ where k is the size of $A = [n] - C$. There is a homeomorphism from D_A to the regular k-gon that sends vertices to vertices, so Proposition 4.2.4, shows that the assertion now follows by applying Lemma 5.6.6 to this k-gon. □

Proposition 5.6.8 (Arbitrary dual parabolics)**.** *Every proper irreducible dual parabolic subgroup of* $\textsc{Braid}_n$ *is equal to the intersection of the maximal irreducible dual parabolic subgroups that contain it and, as a consequence, the collection of of irreducible dual parabolics is closed under intersection. In other words, for all* $n > 0$ *and for every non-empty* $B \subset [n]$,

$$\textsc{Fix}_n(B) = \bigcap_{i \in B} \textsc{Fix}_n(\{i\})$$

and, as a consequence, for all non-empty $C, D \subseteq B$,

$$\textsc{Fix}_n(C \cup D) = \textsc{Fix}_n(C) \cap \textsc{Fix}_n(D).$$

Proof. When B is a singleton, the result is trivial and when B has size 2 both claims are true by Lemma 5.6.6, so suppose that both claims hold for all subsets of size at most k with $k > 1$ and let B be a subset of size $k + 1$. If $i, j \in B$ and $C = B - \{i, j\}$, then $\textsc{Fix}_n(B) = \textsc{Fix}_n(C \cup \{i, j\})$ which is equal to $\textsc{Fix}_n(C \cup \{i\}) \cap \textsc{Fix}_n(C \cup \{j\})$ by Lemma 5.6.7. By applying the second inductive claim to the sets $C \cup \{i\}$ and $C \cup \{j\}$ and simplifying slightly we can rewrite this as $\textsc{Fix}_n(C) \cap \textsc{Fix}_n(\{i\}) \cap \textsc{Fix}_n(\{j\})$. Applying the first inductive claim to the set C shows that first claim holds for B and the second claim for B follows as an immediate consequence. This completes the induction and the proof. □

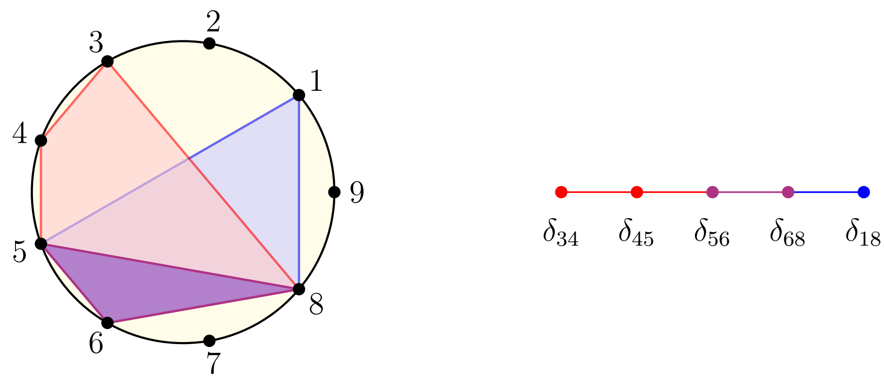

Figure 5.14: Intersections of dual parabolic subgroups of Braid_9, illustrated via convex hulls in the disk and standard parabolic subgroups of Braid_6 - see Example 5.6.9.

Example 5.6.9. Define two subsets of [9] by $A_1 = \{3, 4, 5, 6, 7\}$ and $A_2 = \{1, 5, 6, 8\}$. Then the intersection of the dual parabolic subgroups Braid_{A_1} and Braid_{A_2} in Braid_9 is the dual parabolic $\text{Braid}_{A_1 \cap A_2}$ as indicated in Figure 5.6. To follow the steps in the proof above, let δ_{ij} indicate the positive half-twists about the edge between vertices p_i and p_j. Then the subgroup of Braid_9 with generating set

$$S = \{\delta_{34}, \delta_{45}, \delta_{56}, \delta_{68}, \delta_{18}\}$$

is $\text{Braid}_{A_1 \cup A_2}$, presented as an isomorphic copy of Braid_6. The dual parabolics Braid_{A_1} and Braid_{A_2} are each a standard parabolic subgroup of Braid_6 with respect to this choice of S. That is,

$$\text{Braid}_{A_1} = \langle \delta_{34}, \delta_{45}, \delta_{56}, \delta_{68} \rangle$$

and

$$\text{Braid}_{A_2} = \langle \delta_{56}, \delta_{68}, \delta_{18} \rangle,$$

so by Proposition 2.3.3 their intersection is the subgroup generated by δ_{56} and δ_{68}, so $\textsc{Braid}_{A_1} \cap \textsc{Braid}_{A_2} = \textsc{Braid}_{A_1 \cap A_2}$ as claimed in Proposition 5.6.8.

The algebraic proposition above has a useful topological interpretation. Since $\textsc{Fix}_n(B)$ consists of all braids with the $(b, \cdot)$-strand fixed for all $b \in B$, we may reinterpret the result in the following manner.

Proposition 5.6.10 (Simultaneously fixed strands)**.** *If $\beta \in \textsc{Braid}_n$ has representatives f_1 and f_2 such that $f_1^{b_1}$ and $f_2^{b_2}$ are fixed strands (see Definition 4.2.6 for notation), then there is a representative g of β such that g^{b_1} and g^{b_2} are both fixed. In other words, fixed strands may be fixed simultaneously.*

Notice that this proposition depends on our choice of basepoint in an essential way - fixed strands have this property only when they begin and end in the boundary of the unit disk, since this prevents other strands from wrapping around behind them. A powerful generalization of this idea is presented in Section 7.1, focusing on the notion of strands which "wrap around the outside" of a braid and the resulting subsets of $\textsc{Braid}_n$.

6. The Dual Braid Complex

The central object of study in this dissertation is the *dual braid complex*, first defined by Tom Brady in 2001 [Bra01] (although not by this name). This contractible topological space admits a free action of the braid group and hence the quotient by this action yields a classifying space for the braid group. In this chapter, we exhibit several properties of this complex and related spaces, using the close connections between the dual presentation for the braid group and the dual braid complex.

6.1 The Absolute Order

In this section, we describe a partial order for Sym_n defined by Tom Brady [Bra01].

Definition 6.1.1 (Absolute Order on Sym_n)**.** The conjugacy class of the usual generating set $S = \{\sigma_1, \ldots, \sigma_{n-1}\}$ for Sym_n is the set T of all transpositions. For each $\sigma \in \mathrm{Sym}_n$, define $\ell(\sigma)$ to be the minimum number of transpositions needed to write σ as a product. We refer to ℓ as the *absolute reflection length* on Sym_n and observe its similarity to Definition 5.2.6. Define a relation on Sym_n by declaring that π_1 is covered by π_2 if and only if $\ell(\pi_1) < \ell(\pi_2)$ and $\pi_2 = \pi_1\tau$ for some $\tau \in T$; these covering relations determine a partial order on Sym_n called the *absolute order*. The minimum element is given by the identity permutation, while the $(n-1)!$ cycles of length n form the maximal elements in this poset. More generally, Sym_n is a graded poset with this order and the permutations

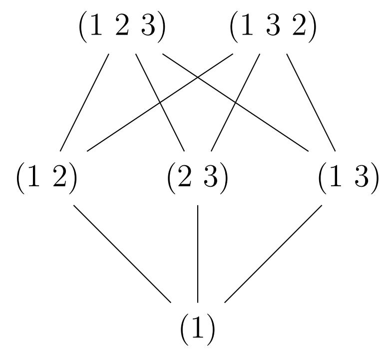

Figure 6.1: The absolute order on the symmetric group Sym_3

of rank k (as defined in Definition 3.1.9) are those which can be expressed as the product of k transpositions and no fewer. That is, the absolute reflection length gives a rank function for this partially ordered set.

The key feature of the absolute order on the symmetric group Sym_n is that each interval between the identity and a maximal element is isomorphic to the noncrossing partition lattice NC_n. Moreover, every maximal element is of height $n-1$ and every element in NC_n is below a maximal element. As such, every interval in this poset is isomorphic to a direct product of smaller noncrossing partition lattices. In the case of the interval from the identity to the n-cycle $\sigma_n = (1 \cdots n)$, we recover our identification of noncrossing partitions with noncrossing permutations from Section 5.1.

Recall from Section 5.3 that the n^{n-2} maximal chains in the noncrossing partition lattice NC_n correspond to the factorizations of δ_n into $n-1$ transpositions. More generally, k-chains in NC_n which begin and end with the minimum and maximum elements correspond to partial factorizations of δ_n.

Also, recall that the positive half-twists in $\textsc{Braid}_n$ form a canonical copy of the set T of transpositions in $\textsc{Sym}_n$. Using this, we define an analogue for absolute reflection length on the braid group.

Definition 6.1.2 (Height on $\textsc{Braid}_n$)**.** Define a homomorphism $h : \textsc{Braid}_n \to \mathbb{Z}$ by declaring $h(\delta_e) = 1$ for each positive half-twist δ_e and extending via the group operation. More generally, if $\beta \in \textsc{Braid}_n$ and

$$\beta = \delta_{e_1}^{m_1} \cdots \delta_{e_k}^{m_k}$$

where each δ_{e_i} is a positive half-twist and $m_i \in \mathbb{Z}$, then the *height* of β is

$$h(\beta) = m_1 + \cdots + m_k,$$

Then if β can be written as a product of positive half-twists, $|h(\beta)|$ is a lower bound on the number of positive half-twists needed to express β as a product. We further observe that h is the abelianization map for $\textsc{Braid}_n$. In particular, notice that h coincides with the absolute reflection length for the dual simple braids. i.e. the sum of the exponents for any decomposition of β into positive half-twists.

The height function on $\textsc{Braid}_n$ appears frequently in literature for the braid group, but usually with other names. In the setting of knot theory, each braid produces a knot via its *braid closure*, and each knot has an associated integer called the *writhe*. It is worth noting that if $\beta \in \textsc{Braid}_n$, then the height of β is the same as the writhe of its braid closure.

Example 6.1.3 (Braid Height). For each $A \subseteq [n]$, the corresponding rotation braid δ_A has height $h(\delta_A) = |A| - 1$. More generally, if δ_π is a dual simple braid with $\pi = \{A_1, \ldots, A_k\}$ a noncrossing partition of $[n]$, then

$$\begin{aligned} h(\delta_\pi) &= h(\delta_{A_1}) + \cdots + h(\delta_{A_k}) \\ &= (|A_1| - 1) + \cdots + (|A_k| - 1) \\ &= n - k \end{aligned}$$

or, in other words, h coincides with the rank function for NC_n given by absolute reflection length.

In the same way that the absolute reflection length gives a rank function for the absolute order on $\textsc{Sym}_n$, the height map gives a rank function for a partial order on $\textsc{Braid}_n$.

Definition 6.1.4 (Absolute Order on $\textsc{Braid}_n$)**.** Define covering relations on $\textsc{Braid}_n$ by declaring that β_1 is covered by β_2 if there is a positive half-twist δ_e on the edge e such that $\beta_2 = \beta_1 \delta_e$. More generally, $\beta_1 < \beta_2$ if we can multiply β_1 by a sequence of positive half-twists to obtain β_2. This induces a partial order on the n-strand braid group called the *absolute order*, and the height map defined above provides a rank function for $\textsc{Braid}_n$ with respect to this order.

The absolute order for $\textsc{Braid}_n$ is compatible with the action of the braid group on itself by left multiplication.

$$
\begin{tikzcd}
\mathrm{HP}\textsc{Braid}_n \arrow[r, hook] \arrow[d, hook] \arrow[dr, hook] & \mathrm{H}\textsc{Braid}_n \arrow[r, two heads, "\textsc{Perm}"] \arrow[d, hook] & \textsc{Alt}_n \arrow[d, hook] \\
\mathrm{P}\textsc{Braid}_n \arrow[r, hook] \arrow[d, two heads, "h"'] & \textsc{Braid}_n \arrow[r, two heads, "\textsc{Perm}"] \arrow[d, two heads, "h"'] \arrow[dr, two heads] & \textsc{Sym}_n \arrow[d, two heads] \\
2\mathbb{Z} \arrow[r, hook] & \mathbb{Z} \arrow[r, two heads] & \mathbb{Z}/2\mathbb{Z}
\end{tikzcd}
$$

Figure 6.2: Horizontal and pure subgroups of the braid group

Proposition 6.1.5. *Let* $\alpha, \beta, \gamma \in \textsc{Braid}_n$ *with* $\alpha \leq \beta$. *Then the intervals* $[\alpha, \beta]$ *and* $[\gamma\alpha, \gamma\beta]$ *are isomorphic in the absolute order for* $\textsc{Braid}_n$.

Proof. First, we claim that the map $\varphi : \textsc{Braid}_n \to \textsc{Braid}_n$ which sends σ to $\varphi(\sigma) = \gamma\sigma$ is order-preserving with respect to the absolute order. If $\alpha \leq \beta$, then there exist positive half-twists $\delta_{e_1}, \ldots, \delta_{e_k}$ such that $\beta = \alpha\delta_{e_1} \cdots \delta_{e_k}$. Hence $\gamma\beta = \gamma\alpha\delta_{e_1} \cdots \delta_{e_k}$, so we also have $\gamma\beta \leq \gamma\alpha$. This is clearly a bijection on $\textsc{Braid}_n$ and the inverse is obtained by multiplying γ^{-1} on the left. Since this map is similarly order-preserving, we see that the intervals $[\alpha, \beta]$ and $[\gamma\alpha, \gamma\beta]$ are isomorphic. □

Unlike in the symmetric group, the identity is not the only braid with a height of zero. For example, if δ_e and $\delta_{e'}$ are distinct positive half-twists, then $\delta_e^{-1}\delta_{e'}$ is a nontrivial braid of height zero. Examples such as this are part of why we refrain from referring to h as the "absolute reflection length on $\textsc{Braid}_n$" - the integer $h(\beta)$ is typically not the minimum number of positive half-twists (and their inverses) needed to express β. Still, the braids of height zero form an interesting subgroup.

Definition 6.1.6 (Horizontal Braid Group)**.** The *horizontal braid group* is the kernel of the height map $h : \textsc{Braid}_n \to \mathbb{Z}$. We may similarly define the *horizontal pure braid group* by taking the kernel of the restriction of h to the pure braid group. Our choice of name is informed by the utility of these groups in the following sections. These subgroups and their corresponding maps fit nicely into a commutative diagram - see Figure 6.1. For more information on interpretations of this group, see [BFW17].

While there is not an obvious geometric presentation for the horizontal braid group, it has a natural geometric appearance in a simplicial complex associated to the braid group - the *dual braid complex.*

6.2 The Dual Braid Complex

The heart of our program to understand the braid group is the construction of three simplicial complexes which arise from the absolute order for $\textsc{Braid}_n$. The first is the *dual braid complex* $\mathcal{D}_n$, a contractible space whose 1-skeleton is the Cayley graph for the n-strand braid group with respect to the set of nontrivial dual simple braids. This space metrically splits as the direct product of $\mathbb{R}$ and the *cross-section complex* $\mathcal{C}_n$. The vertex links of the second complex are all isometric and define the *link complex* $\mathcal{L}_n$. In this section and the following two, we define these complexes and describe many of their properties with explicit examples.

Definition 6.2.1 (Dual Braid Complex)**.** Define a local order on $\textsc{Braid}_n$ by declaring that $\beta_1 \leq \beta_2$ if $\beta_1^{-1}\beta_2$ is a dual simple braid. Since each dual simple braid can be written as a product of positive half twists, it follows that each relation in this local order appears in the absolute order, but not vice versa. The $(n-1)$-dimensional orthoscheme complex $\textsc{Cplx}(\textsc{Braid}_n)$ for this local order is the *dual braid complex* and is denoted $\mathcal{D}_n$ for short. Note that the 1-skeleton of this complex is the (right) Cayley graph for $\textsc{Braid}_n$ in the dual presentation, and if $\beta_1 < \beta_2$ in this local order, then the edge between the vertices labeled by β_1 and β_2 has length $\sqrt{h(\beta_1^{-1}\beta_2)}$. The free action of $\textsc{Braid}_n$ on itself by left multiplication induces a simplicial action on the dual braid complex with the order complex $\Delta(\mathrm{NC}_n)$ as a fundamental domain. Hence, the k-simplices in $\mathcal{D}_n$ have vertex sets of the form $\beta\delta_{\pi_0}, \ldots, \beta\delta_{\pi_k}$ where $\delta_{\pi_0} < \cdots < \delta_{\pi_k}$ is a chain of dual simple braids and $\beta \in \textsc{Braid}_n$. In particular, notice that this action preserves the information in NC_n in the sense that a simplex and its image under the action both project to the same chain of dual simple braids in NC_n. The quotient of this action is a compact Δ-complex with a single vertex which may be viewed as a quotient of the order complex for NC_n.

Example 6.2.2 ($n = 3$). The orthoscheme complex $\Delta(\mathrm{NC}_3)$ is formed by gluing three Euclidean right triangles along their hypotenuses. The dual braid complex $\mathcal{D}_3$ is a contractible simplicial complex with a free geometric action by $\textsc{Braid}_3$, and $\Delta(\mathrm{NC}_3)$ is a fundamental domain for this action. Moreover, the dual braid complex $\mathcal{D}_3$ is homeomorphic to the metric product of a trivalent tree and the real line, although this product is not visible in the simplicial structure. The vertex links of this trivalent tree each consist of only three points, and we may observe that this is precisely the edge link of the long edge in $\Delta(\mathrm{NC}_3)$. See Figure 6.3.

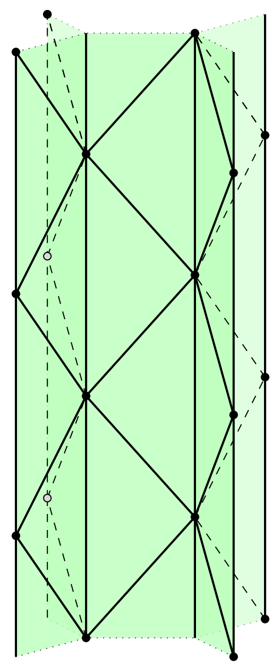

Figure 6.3: A portion of the dual braid complex $\mathcal{D}_3$, as described in Example 6.2.2

As mentioned in Definition 6.2.1, the braid group acts freely and cocompactly on the dual braid complex. Since the action is simplicial and thus preserves edge lengths, notice that $\textsc{Braid}_n$ in fact acts properly discontinuously, cocompactly, and by isometries on $\mathcal{D}_n$. In other words, the braid group acts geometrically on the dual braid complex and thus if the latter is a CAT(0) space, then the former is a CAT(0) group. This is firmly believed to be the case, and was formalized as a conjecture by Brady and McCammond in 2010.

Conjecture 6.2.3 ([BM10])**.** *For every positive integer n, the dual braid complex $\mathcal{D}_n$ is* CAT(0).

In their 2010 paper, Brady and McCammond prove the conjecture when $n \leq 5$ [BM10], which was then extended by Haettel, Kielak, and Schwer to include $n = 6$ in 2016 [HKS16]. The problem remains open for $n > 6$. In Chapter 7, we introduce new tools for tackling the remaining cases.

We may also view the dual braid complex via a quotient of the order complex for NC_n. Each edge in $\Delta(\mathrm{NC}_n)$ corresponds to a 1-chain $\sigma < \tau$ of noncrossing partitions and is labeled by the dual simple braid ${\delta_\sigma}^{-1}\delta_\tau$; define a quotient of the order complex by identifying ordered simplices whose edges have the same orientations and the same labels. The result is a one-vertex Δ-complex (it is not a simplicial complex since its simplices are not determined by their vertex sets). It has n^{n-2} maximal simplices, and the quotient has a loop for each dual simple braid and an oriented 2-simplex for each relation in the dual presentation. In particular, the 2-skeleton of this complex is the presentation complex for the dual presentation for the braid group and the universal cover of this quotient is the dual braid complex.

Example 6.2.4. When $n = 3$, NC_3 is a five-element poset whose order complex consists of three triangles glued along a common edge. The dual braid complex $\mathcal{D}_3$ consists of copies of $\Delta(\mathrm{NC}_3)$ which share edges and vertices, and they combine to form a complex homeomorphic to the direct product of a trivalent tree and $\mathbb{R}$.

More can be said about the homotopy type for $\mathcal{D}_n$; the original definition by T. Brady for the dual braid complex is accompanied by the following theorem, although not phrased in this way.

Theorem 6.2.5 ([Bra01]). *The dual braid complex is contractible and may be written as the direct product of $\mathbb{R}$ and a contractible simplicial complex (see Section 6.3).*

Brady's original complex did not have an associated metric, but the metric given in Definition 6.2.1 was defined by Brady and McCammond in 2010. With this additional information, the theorem above is stronger.

Theorem 6.2.6 ([BM10]). *The dual braid complex is the metric product of $\mathbb{R}$ and a contractible simplicial complex.*

As described above, the braid group acts freely and geometrically on the dual braid complex and the quotient by this action is a finite Δ-complex. By the theorem above, this quotient is a classifying space for the n-strand braid group. Since all classifying spaces for the same group are homotopy equivalent [Hat02], there is a homotopy equivalence between this quotient and the Salvetti complex, although this is not visible via their cell structures.

We close this section by discussing some essential features of the dual braid complex and the appearance of some structures which should be familiar from Section 3.5. It follows from the definition that for each $\beta \in \textsc{Braid}_n$, the elements $\beta, \beta\delta_n, \beta{\delta_n}^2, \ldots$ lie on a copy of $\mathbb{R}$ with edges of length $\sqrt{n-1}$ in $\mathcal{D}_n$. More generally, the maximal simplices in $\mathcal{D}_n$ carry an additional structure which will prove useful to us. One can check that if $\beta_1, \beta_2, \ldots, \beta_{n-1}, \beta_n$ label the vertices of a maximal simplex ordered by increasing height, then $\beta_n = \beta_1\delta_n$ and the vertices $\beta_2, \ldots, \beta_{n-1}, \beta_1\delta_n, \beta_2\delta_n$ form another maximal simplex, and the two share a face of dimension one lower. By repeating this process, we can see that each maximal simplex sits inside a bi-infinite sequence of simplices which forms a connected subcomplex of $\mathcal{D}_n$. In particular, this subcomplex is isometric to a standard column as described in Section 3.5.

Definition 6.2.7 (Maximal Columns)**.** Let $(\beta_1, \ldots, \beta_{n-1})$ be a factorization of δ_n into positive half-twists. Then for each $k \in \mathbb{Z}$, we have $(\beta_1\beta_2 \cdots \beta_{n-1})^k = {\delta_n}^k$ and the braids of this form label vertices which lie on a copy of $\mathbb{R}$ with edges of length $\sqrt{n-1}$ which goes through the vertex labeled by the identity. More generally, the braids obtained by multiplying each β_i in sequence form a bi-infinite sequence with a nice structure in the dual braid complex. Concretely, suppose $a_1, \ldots, a_{n-1}$ are integers satisfying the inequalities $a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq a_1 - 1$ and define $\ell_i = a_i - a_{n-1} \in \{0, 1\}$ for each $i \in [n-2]$. Then define the braid

$$\gamma(a_1, \ldots, a_{n-1}) = {\delta_n}^{a_{n-1}} {\beta_1}^{\ell_1} {\beta_2}^{\ell_2} \cdots {\beta_{n-2}}^{\ell_{n-2}}$$

and observe that $\gamma(m, \ldots, m) = {\delta_n}^m$ for all $m \in \mathbb{Z}$. Moreover, the n braids

$$\gamma(0, 0, \ldots, 0, 0), \gamma(0, 0, \ldots, 0, 1), \gamma(0, 0, \ldots, 1, 1), \ldots, \gamma(0, 1, \ldots, 1, 1), \gamma(1, 1, \ldots, 1, 1)$$

label the vertices of an ordered $(n-1)$-simplex in $\mathcal{D}_n$. By considering all $a_1, \ldots, a_{n-1}$ that satisfy the inequalities above and recalling Definition 3.5.5, this gives an isometric embedding of a column into the dual braid complex. Different choices for our starting factorization yield different columns, all of which intersect in the line spanned by the vertices in $\langle \delta_n \rangle$. The action of $\textsc{Braid}_n$ on itself by right multiplication sends columns to isometric copies; it is not difficult to see that then each maximal simplex in the dual braid complex belongs to a unique column in this sense.

The definition above provides an embedding of a standard column into the dual braid complex for each factorization of δ_n. Partial factorizations of δ_n also correspond to connected subcomplexes of $\mathcal{D}_n$, but they are not columns in the usual sense.

Definition 6.2.8 (Partial Columns)**.** Let $p_\alpha = (\alpha_1, \ldots, \alpha_k)$ be a partial factorization of δ_n and, as above, let $a_1, \ldots, a_{k-1}$ be integers satisfying the inequalties

$$a_1 \geq a_2 \geq \cdots \geq a_{k-1} \geq a_1 - 1,$$

and define $\ell_i = a_i - a_{k-1} \in \{0, 1\}$ for each $i \in [k-2]$. Then we define the braid

$$\gamma(a_1, \ldots, a_{k-1}) = {\delta_n}^{a_{k-1}} {\alpha_1}^{\ell_1} {\alpha_2}^{\ell_2} \cdots {\alpha_{k-2}}^{\ell_{k-2}}.$$

Once again, we obtain a bi-infinite sequence of braids which determine a connected subcomplex of $\mathcal{D}_n$, but in this case the result is not a column in the sense of Definition 3.5.5. Recalling from Definition 5.3.1 the partial order on partial factorizations, consider the maximal factorizations $p_\beta = (\beta_1, \ldots, \beta_{n-1})$ such that $p_\alpha \leq p_\beta$. One can see that the braids in the sequence determined by p_α are obtained by the intersection of all sequences determined by maximal factorizations p_β with $p_\alpha \leq p_\beta$. Hence, the subcomplex determined by these vertices is not a column, but an intersection of columns. Just as a standard column is isometric to the direct product of an $\widetilde{A}_{n-2}$ Coxeter shape and $\mathbb{R}$, this complex is isometric to the direct product of $\mathbb{R}$ and a subsimplex of the $\widetilde{A}_{n-2}$ Coxeter shape. We refer to subcomplexes of this type as *partial columns*, in reference to their correspondence with partial factorizations of δ_n.

It is important to note that while Definition 6.2.7 gives an isometry between certain subcomplexes and a column, this is not the only way columns arise in the dual braid complex. That being said, the subcomplexes of $\mathcal{D}_n$ which are isometric to a column of dimension $n-1$ are precisely those described above. We refer to these as *maximal columns* in the dual braid complex and observe that every maximal simplex of the dual braid complex lies in a unique maximal column.

6.3 The Cross-Section Complex

As described in the preceding section, the action of $\textsc{Braid}_n$ on itself by left multiplication extends to an action of the dual braid complex. It is easy to see that this action sends columns to columns, but $\textsc{Braid}_n$ does not act freely on the set of maximal columns. In particular, left multiplication by the full twist ${\delta_n}^n$ fixes each maximal column and may be viewed as a translation of length $n\sqrt{n-1}$ along the copy of $\mathbb{R}$ mentioned in Theorem 6.2.6.

On the other hand, the *horizontal braid group* (Definition 6.1.6) acts freely on the columns. To see this, recall that the horizontal braid group $\text{H}\textsc{Braid}_n$ consists only of braids β of height $h(\beta) = 0$. Since the height map is a homomorphism, the action of $\text{H}\textsc{Braid}_n$ on the dual braid complex $\mathcal{D}_n$ by left multiplication of the vertex labels preserves height. Hence, since the vertex labels of each column have precisely one element at each height, it is impossible for a horizontal braid to setwise fix the braids labeling a column.

It is worth noting that right multiplication by δ_n on the vertex labels of $\mathcal{D}_n$ yields a cellular map which does not arise from the left action by Braid_n. This map fixes every column setwise and translates by $\sqrt{n-1}$. Iterating this, we observe that right multiplication by ${\delta_n}^n$ *does* arise as a left action since ${\delta_n}^n$ is in the center of Braid_n.

As we described in Section 3.5, each column decomposes as the metric direct product of a Euclidean simplex and a copy of $\mathbb{R}$. Then the maximal columns in the dual braid complex each exhibit this direct product structure, and these splits are compatible in the following sense. The height map $h : \text{Braid}_n \to \mathbb{Z}$ extends to a Morse function $h : \mathcal{D}_n \to \mathbb{R}$ (in the sense of Bestvina-Brady, since $\mathcal{D}_n$ is not a manifold [BB97]) which is the projection map for a decomposition of the dual braid complex into the direct product of $\mathbb{R}$ and a simplicial complex we call the *cross-section complex*, and restricting the domain of h to a maximal column gives the expected projection map for its direct product structure. Moreover, the codomain restriction of h to $\mathbb{Z}$ returns the absolute reflection length defined above.

Definition 6.3.1 (Cross-Section Complex)**.** Let $n \geq 2$ and consider the dual braid complex $\mathcal{D}_n$ with dimension $n-1$. Then $\langle \delta_n \rangle \cong \mathbb{Z}$ labels a sequence of vertices in the dual braid complex which share a copy of $\mathbb{R}$ with edges of length $\sqrt{n-1}$. Define an abstract simplicial complex with vertices labeled by the cosets in $\text{Braid}_n/\langle \delta_n \rangle$ and a maximal $(n-2)$-simplex on each set of $n-1$ vertices with representatives $\beta_1\langle \delta_n \rangle, \ldots, \beta_{n-1}\langle \delta_n \rangle$

such that $\beta_1, \ldots, \beta_{n-1}, \beta_1\delta_n$ label the vertices of a $n-1$-simplex in $\mathcal{D}_n$. This produces a simplicial complex of dimension $n-2$ which we call the *cross-section complex* $\mathcal{C}_n$, where each maximal simplex in $\mathcal{C}_n$ corresponds to a unique maximal column in the dual braid complex $\mathcal{D}_n$; we may then identify this complex with $h^{-1}(0)$, where h is the Morse function described above. In other words, the metric product of the cross-section complex $\mathcal{C}_n$ with $\mathbb{R}$ is isometric to the dual braid complex $\mathcal{D}_n$, but with a different cell structure [Bra01]. Recalling our description of columns in Definition 3.5.5, we may identify each maximal column of $\mathcal{D}_n$ with a column in $\mathbb{R}^{n-1}$ and, according to our identification of $\mathcal{C}_n$ with $h^{-1}(0)$, give each maximal simplex in the cross-section complex the metric of an $\widetilde{A}_{n-2}$ Coxeter shape. This provides a piecewise Euclidean metric for the cross-section complex which is precisely the metric inherited from the dual braid complex.

The cross-section complex is an example of a far more general phenomenon, demonstrated for spherical Artin groups by Mladen Bestvina [Bes99] and for the more general class of Garside groups by Ruth Charney, John Meier, and Kim Whittlesey [CMW04]. They define a similar type of cross-section complex in these settings, but without the piecewise Euclidean metric considered here.

Notice that the action of $\textsc{Braid}_n$ on the dual braid complex descends to an action on the cross-section complex, but one which is no longer properly discontinuous since each vertex has infinite stabilizer. Restricting to an action by the horizontal braid group $\textsc{HBraid}_n$ resolves this issue and we obtain a geometric action on the cross-section complex, although we remark that this action is no longer transitive on the vertices. As remarked earlier in the section, the horizontal braid group acts on the dual braid complex in a height-preserving fashion. There are then $n-1$ orbits for the action of $\mathrm{HB}\textsc{raid}_n$ on the cross-section complex $\mathcal{C}_n$ since each coset $\beta\langle\delta_n\rangle$ has a unique representative with height in the set $\{1,\ldots,n-1\}$.

Alternatively, let Z_n be the center of $\textsc{Braid}_n$ - this is an infinite cyclic subgroup of $\textsc{Braid}_n$, generated by ${\delta_n}^n$. Then the quotient $\textsc{Braid}_n/Z_n$ acts geometrically on the cross-section $\mathcal{C}_n$ with finite stabilizer. This action is related to the action by $\mathrm{HB}\textsc{raid}_n$ in the sense that the horizontal braid group injects into $\textsc{Braid}_n/Z_n$ with an image of finite index.

6.4 The Noncrossing Partition Link

The vertex links of the cross-section complex may each be identified with links of edges in the dual braid complex which are labeled by δ_n. These vertex links are all isometric and their unique isometry type is a structure of special interest.

Since each vertex σ in the cross-section complex is the retraction of a vertical copy of $\mathbb{R}$ in the dual braid complex, the vertex link of σ in $\mathcal{C}_n$ is isometric to the link of the edge in $\mathcal{D}_n$ between vertices $\mathbf{v}_\sigma$ and $\mathbf{v}_{\sigma\delta_n}$. In particular, every vertex link in the cross-section complex is isometric to the link complex for the noncrossing partition lattice.

Definition 6.4.1 (Noncrossing Partition Link)**.** The *noncrossing partition link* (or *link complex*) is the $(n-2)$-dimensional PS complex given by the poset link $\textsc{Link}(\mathrm{NC}_n)$ as described in Definition 3.3.6. For convenience and to match the notation of our other complexes, we denote $\textsc{Link}(\mathrm{NC}_n)$ by the shorthand $\mathcal{L}_n$. As the link of a piecewise-Euclidean orthoscheme complex, the noncrossing partition link is a piecewise-spherical complex where each simplex is a Coxeter shape of type A_{n-2}. For example, the A_2 Coxeter shape is a circular arc of length $\frac{\pi}{3}$ and the A_3 Coxeter shape is a spherical triangle with edge lengths $\frac{\pi}{2}$, $\frac{\pi}{3}$, and $\frac{\pi}{3}$. The n^{n-2} maximal simplices in the link complex are labeled by the factorizations of δ_n into $n-1$ positive half-twists, or alternatively by the maximal chains of NC_n. More generally, k-simplices in the noncrossing partition link correspond to partial factorizations of δ_n into $k+2$ dual simple braids.

Since the vertex links of the cross-section complex are each isometric to the noncrossing partition link, the Link Condition (Proposition 3.4.13) tells us that if $\mathcal{L}_n$ is CAT(1), then $\mathcal{C}_n$ is CAT(0) and hence the dual braid complex $\mathcal{D}_n$ is CAT(0). It is strongly believed that the noncrossing partition link is CAT(1), and recent attempts to prove that the braid group is CAT(0) have followed this approach ([BM10], [HKS16]). For the remainder of this section, we review some of the evidence which suggests that the link complex is CAT(1).

The noncrossing partition link is closely connected with a *spherical building*, a type of piecewise-spherical metric space with close ties to finite Coxeter groups and an exceptional degree of symmetry. While we survey the broad ideas involved, a complete treatment may be found in the essential reference by Abramenko and Brown [AB08].

Definition 6.4.2 (Buildings)**.** A simplicial complex Δ is a *building* if it can be expressed as a union of subcomplexes called *apartments* which satisfy the following three properties:

1. Each apartment is a Coxeter complex.

2. Any two simplices in Δ are contained in a common apartment.

3. If Σ and Σ' are two apartments which each contain the simplices A and B, then there is a simplicial homeomorphism $\Sigma \to \Sigma'$ which fixes A and B pointwise.

As a consequence of the properties above, each apartment is a Coxeter complex for a fixed Coxeter group W; when W is finite, we say that Δ is a *spherical building.* In this scenario, Δ inherits a piecewise spherical metric from the spherical Coxeter complex of the associated type and the resulting metric space is known to be CAT(1) - see [BH99], Theorem 10A.4.

We need only one example of a building, given by the geometry of the subspaces for a fixed vector space.

Definition 6.4.3 (Linear Subspace Poset)**.** For each field $\mathbb{F}$ and nonnegative integer n, the *linear subspace poset* $L_n(\mathbb{F})$ is the set of all linear subspaces of the vector space $\mathbb{F}^n$, ordered by inclusion. The associated poset link has simplices labeled by chains of nonzero proper subspaces in $\mathbb{F}^n$ and is known to be a spherical building of type A_{n-1}. In particular, the poset link for $L_n(\mathbb{F})$ is a CAT(1) metric space.

Our use for the linear subspace poset comes from the fact that there is an embedding of NC_n into $L_{n-1}(\mathbb{F}_2)$, first described by Brady and McCammond [BM10]. Recall that the noncrossing partition lattice NC_n embeds in the lattice Π_n of all partitions via inclusion. Each partition of $[n]$ determines a linear subspace of ${\mathbb{F}_2}^n$ by sending each block to a hyperplane whose defining equation is determined by setting equal to zero the sum of the coordinates indexed by the block. For example, the partition $\{\{1,2,4\},\{3,6\},\{5\}\}$ is sent to the subspace of ${\mathbb{F}_2}^6$ determined by the equations $x_1+x_2+x_4=0$, $x_3+x_6=0$, $x_5=0$. Hence, there is an order-preserving map from NC_n into the linear subspace poset $L_n(\mathbb{F}_2)$. Since each subspace in the image of this map is contained in the $(n-1)$-dimensional hyperplane determined by the equation $x_1+\cdots+x_n=0$, we can restrict this to an order-preserving map $\mathrm{NC}_n \to L_{n-1}(\mathbb{F}_2)$. Moreover, the link complex $\mathcal{L}_n = \textsc{Link}(\mathrm{NC}_n)$ embeds into $\textsc{Link}(L_{n-1}(\mathbb{F}_2))$. For more information on these embeddings and others of similar types, see the recent work by Julia Heller and Petra Schwer [HS17].

The embedding above may be used to import a piecewise-spherical metric from the spherical building back to the noncrossing partition link, and this metric matches the one induced by taking the link of an edge in the orthoscheme complex for the noncrossing partition lattice. While we do not make use of this connection with spherical buildings, it suggests that the metric information contained in the noncrossing partition link is the correct one to examine.

On the other hand, the combinatorics of the noncrossing partition link give us a way of labeling simplices in the cross-section complex. Since each vertex link in the cross-section $\mathcal{C}_n$ may be identified with the link complex $\mathcal{L}_n$, we may fix a vertex and label each incident simplex by a factorization of δ_n.

The dual braid complex, the cross-section complex, and the noncrossing partition link each highlight different aspects of the dual braid group and will each be useful for

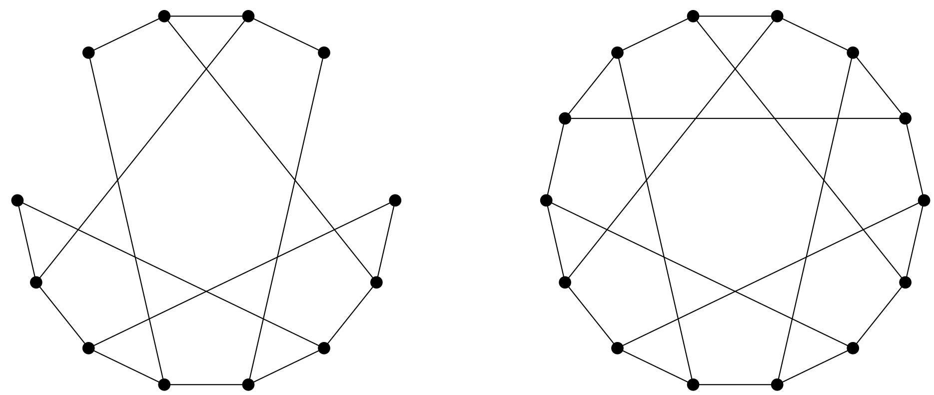

Figure 6.4: The link complex $\mathcal{L}_4$ and the spherical building in which it embeds.

certain purposes. Moreover, it is worth noting that they are each a different dimension: $\mathcal{D}_n$ is $(n-1)$-dimensional, $\mathcal{C}_n$ is $(n-2)$-dimensional, and the link complex $\mathcal{L}_n$ is $(n-3)$-dimensional. The relationships between these complexes will allow us to prove several facts about the braid group and its associated spaces.

We conclude the section with two examples in low dimensions. The first is a restatement of Example 6.2.2 with the tools developed in this chapter.

Example 6.4.4 ($n = 3$). When $n = 3$, the noncrossing partition lattice NC_3 contains 5 elements and its orthoscheme complex $\Delta(\mathrm{NC}_3)$ consists of three right isoceles triangles glued along their hypotenuses. The dual braid complex $\mathcal{D}_3$ then has this orthoscheme complex as a fundamental domain with columns which are homeomorphic to the direct product of a unit interval (the $\widetilde{A}_1$ Coxeter shape) and $\mathbb{R}$. Moreover, the cross-section complex $\mathcal{C}_3$ is a trivalent tree and thus $\mathcal{D}_3$ is metrically the direct product of this tree and $\mathbb{R}$. Since the cross-section complex is a graph, the link complex is just a collection of points; in this case $\mathcal{L}_3$ is just three points, which is evident from the fact that NC_3 without its bounding elements has only three vertices and no larger simplices.

Example 6.4.5 ($n = 4$). When $n = 4$, the noncrossing partition lattice NC_4 contains 14 elements and has $4^2 = 16$ maximal chains - see Figure 5.4. Hence, the order complex $\Delta(\mathrm{NC}_4)$ has 16 3-orthoschemes, although the complex as a whole is more difficult to envision. Each 3-orthoscheme in the dual braid complex $\mathcal{D}_4$ fits into a column which is isometric to the product of an equilateral triangle (the $\widetilde{A}_2$ Coxeter shape) and $\mathbb{R}$, and hence the cross-section complex $\mathcal{C}_4$ is built out of equilateral triangles. In particular, the 16 triangles incident to each vertex in $\mathcal{C}_4$ are recorded in the link complex $\mathcal{L}_4$, which consists of 16 edges and 12 vertices - see Figure 6.4.

6.5 Dual Parabolic Subcomplexes

As described in Section 5.6, the dual parabolic subgroups of $\textsc{Braid}_n$ provide a rich algebraic structure which corresponds to topological structure in the complexes described in the last three sections. In this section, we describe subcomplexes of the dual braid complex, the cross-section complex, and the link complex which correspond to the dual parabolic subgroups and discuss the relationships between these settings.

To begin, we consider the subcomplex of the dual braid complex induced by a dual parabolic subgroup $\textsc{Braid}_A \subseteq \textsc{Braid}_n$.

Definition 6.5.1 (Dual Parabolic Subcomplexes)**.** Let $A \subseteq [n]$ with $|A| = k$. Then as described in Definition 5.2.13, DS_A is the subposet of DS_n which is defined by the interval $[\hat{0}, \delta_A]$, and we again observe that DS_A is isomorphic to DS_k. The *dual parabolic subcomplex* $\textsc{Cplx}(\textsc{Braid}_A)$ is the simplicial complex with vertex set indexed by the elements of the dual parabolic subgroup $\textsc{Braid}_A$ and an ℓ-simplex on vertices $\beta_0, \ldots, \beta_\ell$ when ${\beta_i}^{-1}\beta_j \in \mathrm{DS}_A$ for all $0 \leq i < j \leq \ell$. Similar to the case for the entire dual braid complex $\mathcal{D}_n$, we denote $\textsc{Cplx}(\textsc{Braid}_A)$ as $\mathcal{D}_A$ and we may define this subcomplex abstractly as the orthoscheme complex for an appropriate local order on $\textsc{Braid}_A$. The 1-skeleton of this complex forms the right Cayley graph for the dual parabolic subgroup $\textsc{Braid}_A$ with respect to the generating set DS_A^*, the nontrivial elements of DS_A. More generally, it is an easy exercise to see that $\mathcal{D}_A$ is a $(k-1)$-dimensional complex which is isometric to $\mathcal{D}_k$ and has a natural embedding into the dual braid complex $\mathcal{D}_n$ which is induced by the inclusion of $\textsc{Braid}_A$ into $\textsc{Braid}_n$.

Example 6.5.2. Let $A = \{1, 2, 3\}$ be a subset of $[4]$. Then DS_A is an isomorphic copy of DS_3 within DS_4 and the dual parabolic subcomplex $\mathcal{D}_A$ is the image of an isometric embedding of $\mathcal{D}_3$ into $\mathcal{D}_4$. In other words, $\mathcal{D}_3$ is the metric product of a trivalent tree and $\mathbb{R}$, embedded in the dual braid complex $\mathcal{D}_4$. It is worth noting that while this copy of $\mathbb{R}$ corresponds to the "vertical" direction in $\mathcal{D}_A$ obtained via multiplication by δ_A, this is not the same as the vertical direction in $\mathcal{D}_4$. As we prove in Proposition 6.5.5, the projection of $\mathcal{D}_A$ to the cross-section complex $\mathcal{C}_4$ is a homeomorphism.

In the same way that the dual parabolic subgroups of $\textsc{Braid}_n$ have well-understood intersections, the corresponding dual parabolic subcomplexes of $\mathcal{D}_n$ have a pleasing structure in their intersections.

Proposition 6.5.3 (Intersections of Dual Parabolic Subcomplexes). *Let A_1 and A_2 be subsets of $[n]$. Then*

$$\mathcal{D}_{A_1} \cap \mathcal{D}_{A_2} = \mathcal{D}_{A_1 \cap A_2}$$

or, in other words, the dual parabolic subcomplexes of $\mathcal{D}_n$ intersect in smaller dual parabolic subcomplexes.

Proof. Let $\beta_0, \ldots, \beta_\ell \in \textsc{Braid}_n$ label the vertices of an ℓ-simplex in $\mathcal{D}_n$. Then this simplex lies in the intersection $\mathcal{D}_{A_1} \cap \mathcal{D}_{A_2}$ if and only if $\beta_i^{-1}\beta_j$ is in both NC_{A_1} and NC_{A_2} for all $0 \leq i < j \leq \ell$. By Proposition 5.2.16 we know that $\mathrm{NC}_{A_1} \cap \mathrm{NC}_{A_2} = \mathrm{NC}_{A_1 \cap A_2}$, so the ℓ-simplex in question lies in $\mathcal{D}_{A_1 \cap A_2}$. □

Example 6.5.4. Let $n = 4$ and define $A_1 = \{1, 2, 3\}$ and $A_2 = \{1, 2, 4\}$. Then $\mathcal{D}_{A_1}$ and $\mathcal{D}_{A_2}$ are each isometrically embedded copies of $\mathcal{D}_3$ within the dual braid complex $\mathcal{D}_4$. The intersection of NC_{A_1} and NC_{A_2} is $\mathrm{NC}_{\{1,2\}}$, a totally ordered set consisting of two elements. The corresponding subcomplex of $\mathcal{D}_4$ is then a copy of $\mathbb{R}$ which lies in each of the two copies of the product of a trivalent tree and $\mathbb{R}$, given by $\mathcal{D}_{A_1}$ and $\mathcal{D}_{A_2}$.

Notice that the isometrically embedded copies of $\mathcal{D}_k$ in $\mathcal{D}_n$ described above are each of codimension $n - k$. While these provide a useful recursive structure to the dual braid complex, this "lack" of dimension prevents the dual parabolic subcomplexes from being useful in understanding the curvature of $\mathcal{D}_n$. To "fix" this, we expand this subcomplex into one of the full dimension for the dual braid complex by exploring a generalization of dual parabolic subgroups. In particular, we generalize the notion of a strand being fixed - see Chapter 7 for a discussion of these results.

The analogues for dual parabolic subcomplexes in both the cross-section complex and the link complex have individually interesting structures which are worth recording.

Proposition 6.5.5 (Dual Parabolics in the Cross-Section Complex)**.** *Define $p : \mathcal{D}_n \to \mathcal{C}_n$ to be the projection map obtained by deformation retracting the dual braid complex to the points of height zero, and fix a proper subset $A \subseteq [n]$. Then the restriction of p to the dual parabolic subcomplex $\mathcal{D}_A$ is a simplicial homeomorphism onto its image $p(\mathcal{D}_A)$.*

Proof. We begin by showing that the restriction of p to the vertices of $\mathcal{D}_A$ is an injection. If $\beta_1, \beta_2 \in \textsc{Braid}_A$ label vertices which are sent to the same vertex via the projection map, then $\beta_2 = \beta_1 \delta^k$ for some integer k. Hence $\beta_1^{-1}\beta_2 = \delta^k$ and thus $\delta^k \in \textsc{Braid}_A$, which is a contradiction unless $k = 0$ since every nonzero power of δ fixes none of the strands with labels in $[n] - A$. Therefore, we must have $\beta_1 = \beta_2$, so this restriction of p is injective.

Since the vertices of each dual parabolic subcomplex map injectively into the cross-section complex and each simplex is determined by its vertices, the proof is complete. $\square$

It is worth noting that, while the projection of the dual braid complex $\mathcal{D}_n$ onto the cross-section complex $\mathcal{C}_n$ is well-understood, the metric on parabolic subcomplexes is distorted by this map. Specifically, orthoschemes in $\mathcal{D}_A$ are sent to Coxeter shapes in the projection. Despite this, there is a lot that we know about the metric for these projected subcomplexes.

The vertex links in the cross-section complex are each isometric to $\mathcal{L}_n$, so the projection of $\mathcal{D}_A$ into the cross-section complex has vertex links which are isometric to subcomplexes of the link complex.

Proposition 6.5.6 (Dual Parabolics have Isometric Links)**.** *Let $A \subseteq [n]$ be a proper subset. Then the dual parabolic subcomplex $\mathcal{D}_A$ embeds homeomorphically into the cross-section complex $\mathcal{C}_n$ and the vertex links in the image are all isometric subcomplexes of $\mathcal{L}_n$.*

Proof. First, note that since the projection map from the dual braid complex to the cross-section complex restricts to a simplicial homeomorphism on $\mathcal{D}_A$, then the vertex links in the image of this map are subcomplexes of the vertex links in $\mathcal{C}_n$. In other words, they are subcomplexes of the noncrossing partition link $\mathcal{L}_n$. It then suffices to show that each vertex link is isometric to that of the identity braid. Consider a vertex in this image with label the coset representative $\beta\langle\delta_n\rangle$, where $\beta \in \textsc{Braid}_A$. Then the action on the dual braid complex given by multiplying β^{-1} on the right is an isometry which stabilizes $\mathcal{D}_A$ and carries β to the identity. This map restricts to an isometry on the cross-section complex which carries the vertex $\beta\langle\delta_n\rangle$ to $\langle\delta_n\rangle$, which then descends to an isometry of the two vertex links. $\square$

The isometric vertex links of $p(\mathcal{D}_A)$ are determined by a simple combinatorial condition on the link complex $\mathcal{L}_n$.

Definition 6.5.7 (Dual Parabolics in the Link Complex)**.** Let $A \subseteq [n]$. Each simplex in $\mathcal{L}_n$ is labeled by a partial factorization of δ_n; define $p(\mathcal{L}_A)$ to be the subcomplex determined by the simplices labeled by partial factorizations which contain the dual simple braid ${\delta_A}^{-1}\delta_n$. This notation is meant to be suggestive of the fact that we can think of $\mathcal{L}_n$ as the link of an edge in $\mathcal{D}_n$, obtained via the intersection of $\mathcal{D}_n$ with an $(n-2)$-sphere of radius ϵ centered at the midpoint of that edge. With this picture, the projection p into the cross-section carries this $(n-2)$-sphere homeomorphically to a vertex link in the cross-section, and by restricting which simplices of $\mathcal{L}_n$ are projected, we obtain the vertex link for a subcomplex of the cross-section.

Before exploring some properties of these subcomplexes, we justify our suggestive notation.

Proposition 6.5.8. *Let $A \subseteq [n]$. Then each vertex link in $p(\mathcal{D}_A)$ is isometric to $p(\mathcal{L}_A)$.*

Proof. By Proposition 6.5.6, it suffices to consider the link of the identity vertex in $\mathcal{C}_n$, i.e. the vertex labeled by $\langle \delta_n \rangle$. By Proposition 6.5.5, the simplices in $p(\mathcal{D}_A)$ incident to $\langle \delta_n \rangle$ correspond to simplices in $\mathcal{D}_A$ incident to the identity.

Suppose $|A| = k$ and $\beta_1, \ldots, \beta_{k-1} \in \textsc{Braid}_A$ label the vertices of a $(k-2)$-simplex in $\mathcal{D}_A$ in increasing height (i.e. a maximal simplex in this subcomplex) such that $\beta_i = e$ for some i. Since this is a maximal simplex in $\mathcal{D}_A$, $\beta_{k-1} = \beta_1 \delta_A$ and thus the vertices $\beta_1, \ldots, \beta_{k-1}, \beta_{k-1}{\delta_A}^{-1}\delta_n$ label a $(k-1)$-simplex in $\mathcal{D}_n$ which corresponds to a partial factorization of δ_n. As described in Definition 6.2.8, these vertices belong to a bi-infinite sequence which determines a partial column incident to the identity. We may then select a string of k consecutive elements from this list which begins at $\beta_i = e$ and thus labels a simplex in our chosen partial column which is based at the identity. The corresponding simplex in $\mathcal{L}_n$ is then labeled by a partial factorization which contains ${\delta_A}^{-1}\delta_n$. We have therefore shown that every simplex in the vertex link of $\langle\delta_n\rangle$ contains ${\delta_A}^{-1}\delta_n$ and thus belongs to $p(\mathcal{L}_A)$.

We now consider the other direction. Since $p(\mathcal{L}_A)$ is defined to be the subcomplex determined by partial factorizations of δ_n which include ${\delta_A}^{-1}\delta_n$, we may easily construct a simplex based at the identity in $\mathcal{D}_A$ which corresponds to each such partial factorization. The projection to $p(\mathcal{D}_A)$ then provides a simplex in the cross-section which includes the vertex labeled by $\langle\delta_n\rangle$. Hence, the partial factorization we began with labels a simplex in the vertex link of the identity in $p(\mathcal{D}_A)$. $\square$

Unlike the case for the dual braid complex and the cross-section complex, $p(\mathcal{L}_A)$ is not homeomorphic to $\mathcal{L}_{|A|}$. Thankfully, however, this is not too far from the truth.

Proposition 6.5.9. *Let $A \subseteq [n]$ with $|A| = k$. Then $p(\mathcal{L}_A)$ is homeomorphic to the suspension of $\mathcal{L}_k$.*

Proof. By Proposition 6.5.5, $\mathcal{D}_A$ is homeomorphic to its image $p(\mathcal{D}_A) \subseteq \mathcal{C}_n$ and thus its vertex links are homeomorphic as well. Since $\mathcal{D}_A$ is isometric to $\mathcal{D}_k$, it suffices to consider the vertex links in this complex. Then $\mathcal{D}_k$ splits as the direct product of $\mathbb{R}$ and $\mathcal{C}_k$, so by Definition 3.3.8, the link of a vertex in $\mathcal{D}_k$ is simply the spherical join of the vertex links in $\mathbb{R}$ (with vertices in $\langle\delta_k\rangle$) and $\mathcal{C}_k$. Since the vertex links are copies of $\mathbb{S}^0$ in the former and the $\mathcal{L}_k$ in the latter, we're done. $\square$

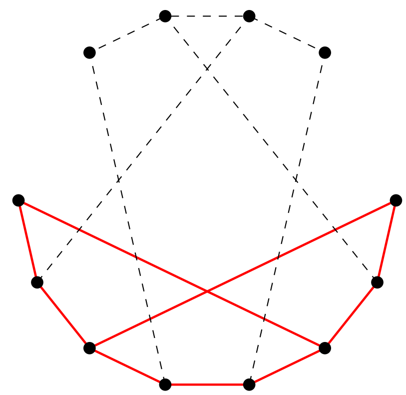

Figure 6.5: The link complex $\mathcal{L}_4$ and the subcomplex labeled by factorizations including $\delta_{\{1,4\}}$. The subcomplex is homeomorphic to the suspension of $\mathcal{L}_3$ and its complement is contractible.

Example 6.5.10. Let $n = 4$ and $A = \{1, 2, 3\}$. Then ${\delta_A}^{-1}\delta_n = \delta_{\{1,4\}}$ and there are nine factorizations of δ_n which include $\delta_{\{1,4\}}$. These nine factorizations label a subcomplex of $\mathcal{L}_4$ which is homeomorphic to the suspension of three dots, i.e. the link complex $\mathcal{L}_3$. See Figure 6.5 for an illustration.

In the case when $|A| = n-1$, the link complex is decomposed into two pieces, each of which has easily understandable homotopy type. A proof of the following theorem may be found in [DM].

Theorem 6.5.11 ([DM]). *Let $A = [n] - \{s\}$ and define δ_s to be the positive half-twist which swaps $\boldsymbol{v}_{s-1}$ and $\boldsymbol{v}_s$. Then the maximal simplices of $\mathcal{L}_n$ may be partitioned into two sets: those whose corresponding factorizations of δ_n contain δ_s, and those which do not. Each set determines a subcomplex of $\mathcal{L}_n$: the former is $p(\mathcal{L}_A)$, homeomorphic to the suspension of $\mathcal{L}_{n-1}$, and the latter is contractible.*

7. Braids with Boundary-Parallel Strands

As described in Section 5.6, the dual parabolic subgroups of BRAID_n may be viewed as collections of braids obtained by specifying a subset of the strands to be fixed. A natural generalization of this is obtained by instead selecting strands which are confined to the boundary of D_n, the canonical n-gon we use to represent braids. In this chapter we discuss the notion of boundary-parallel strands, i.e. those which "wrap around the outside" of a braid. This characterization for braids appears in forthcoming research by the author with Jon McCammond and Stefan Witzel [DMW], where it is used to describe several subcomplexes of the dual braid complex and analyze their curvature. The result is a significant improvement in our understanding of the curvature for the dual braid complex.

7.1 Boundary Braids

We begin by defining what it means for a braid to have *boundary-parallel* strands before examining the basic properties which follow. Recall that D_n is a regular convex n-gon with vertex set $P_n = \{p_1, \ldots, p_n\}$ and that think of BRAID_n as the fundamental group $\pi_1(\text{UCONF}_n(D_n), P_n)$. Individual braids may then be represented by motions in D_n of the n points in P_n, leading to the following new characterization of certain braids.

Definition 7.1.1 (Boundary Braids)**.** Let β be an n-strand braid and let $b, c \in [n]$. If β has a representative f which moves p_b to p_c and the image of the strand f^b_c remains in the boundary of D_n, then we say that the (b, c)-strand of f is *boundary-parallel* and that β is a (b, c)*-boundary braid.* If c or b is unspecified, we refer to β as a $(b, \cdot)$*-boundary braid* or a $(\cdot, c)$*-boundary braid*, respectively. More generally, if $B \subseteq [n]$ and f^b is boundary-parallel for each $b \in B$, we say that the $(B, \cdot)$-strands of f are *simultaneously boundary-parallel.* If β has such a representative and the vertices in P_B are sent bijectively by this representative to P_C, we say that β is a (B, C)*-boundary braid.* In the case that C or B is unspecified, we refer to β as a $(B, \cdot)$*-boundary braid* or a $(\cdot, B)$*-boundary braid*, respectively. We denote the set of all (B, C)-boundary braids by $\textsc{Braid}_n(B, C)$, which is naturally contained in both the set of all $(B, \cdot)$-boundary braids and the set of all $(\cdot, C)$-boundary braids, denoted $\textsc{Braid}_n(B, \cdot)$ and $\textsc{Braid}_n(\cdot, C)$ respectively. Notice that these are not subgroups of $\textsc{Braid}_n$ unless $B = C$, as they fail to be closed under the group operation and do not contain their inverses. However, they do form a type of subgroupoid: if $\beta_1 \in \textsc{Braid}_n(B, C)$ and $\beta_2 \in \textsc{Braid}_n(C, D)$, then $\beta_1\beta_2 \in \textsc{Braid}_n(B, D)$ and ${\beta_1}^{-1} \in \textsc{Braid}_n(C, B)$.

Braids with boundary-parallel strands are not uncommon - in fact, many of our usual examples (including all dual simple braids) have at least one boundary-parallel strand. For rotation braids, the strands that are boundary-parallel in the standard representative are the only boundary-parallel strands in any representative.

Lemma 7.1.2 (Rotation Braids)**.** *Let $A \subseteq [n]$ and define*

$$B = \{b \mid b \notin A \text{ or both } b, b+1 \in A\} \subseteq [n].$$

Then δ_A is a $(B, \cdot)$-boundary braid but not an $(b, \cdot)$-boundary braid for any $b \in [n] - B$.

Proof. The first statement is straightforward by considering the standard representative of δ_A given by constant-speed parametrization of each strand along the boundary of the subdisk D_A.

For the second statement, let $b \in [n]-B$. Then $b \in A$ but $b+1 \notin A$. Fix the standard representative f of δ_A described above and consider the topological disk in $I \times D_n$ which is bounded by the $(b-1, \cdot)$- and $(b+1, \cdot)$- strands of f, together with the line segments connecting p_{b-1} and p_{b+1} in each of $\{0\} \times D_n$ and $\{1\} \times D_n$. Since $b+1 \notin A$, the f^{b+1} strand is fixed. If $b-1 \notin A$, then the f^{b-1} strand is also fixed. If $b-1 \in A$, then the $(b-1, \cdot)$ strand terminates at the vertex p_b. In both cases, the strand f^b starts on one side of the disk and ends on the other side, and thus it transversely intersects the disk an odd number of times. Since the parity of the number of transverse intersections is preserved under homotopy of strands and strands which remain in the boundary have no such intersections, we may conclude that the $(b, \cdot)$-strand is not boundary-parallel in any representative for δ_A. □

Recall that each dual simple braid $\delta_\pi \in \mathrm{DS}_n$ may be written as a product of rotation braids $\delta_\pi = \delta_{A_1} \cdots \delta_{A_k}$, where the A_i are disjoint subsets of $[n]$. By Lemma 7.1.2, δ_π is a $(b, \cdot)$ boundary braid if and only if either $b \in [n] - (A_1 \cup \cdots \cup A_k)$ or both b and $b+1$ are contained in A_i for some i. Comparing this condition with Definition 5.4.2, we see that $\mathrm{DS}_n(B,\cdot)$ is precisely the set of dual simple braids which are $(B,\cdot)$-boundary braids.

Remark 7.1.3. Recall that the pure braid group $\mathrm{PB}\textsc{raid}_n$ is the kernel of the map $\textsc{Braid}_n \twoheadrightarrow \textsc{Sym}_n$ which sends each braid to the permutation induced by its strands. We may then define $\mathrm{PB}\textsc{raid}_n(B)$ to be the intersection of $\mathrm{PB}\textsc{raid}_n$ and $\textsc{Braid}_n(B,B)$ and remark that this *is* a subgroup of $\mathrm{PB}\textsc{raid}_n$. In fact, one can show that if $|B| = k$, then $\mathrm{PB}\textsc{raid}_n(B)$ is isomorphic to the direct product $\mathrm{PB}\textsc{raid}_{n-k} \times \mathbb{Z}$.

Following Definition 7.1.1, it is natural to wonder: are there any braids with boundary-parallel strands which are not *simultaneously* boundary-parallel? The remainder of this section is dedicated to answering this question in the negative.

Definition 7.1.4 (Wrapping number)**.** Let β be a $(b,\cdot)$-boundary braid and let f be a representative for which the image of f^b lies in the boundary of D_n. If we view the boundary of D_n as an n-fold cover of the standard cell structure for $\mathbb{S}^1$ with one vertex and one edge, then boundary paths in ∂D_n that start and end at vertices of D_n may be considered as lifts of loops in $\mathbb{S}^1$. More concretely, let $\varphi : \mathbb{R} \to \partial D_n$ be a covering map such that $\varphi(i) = p_i$ for each $i \in \mathbb{Z}$. Let $\tilde{f}^b$ be any lift of f^b via this covering and define the *wrapping number* of the (b,c)-strand of f to be $w_f(b,c) = \tilde{f}^b(1) - \tilde{f}^b(0)$. Since f^b lies in the boundary of the disk D_n, we may consider it as a map $f^b : [0,1] \to \mathbb{S}^1$. If $\mathbb{S}^1$ is identified with the unit circle in $\mathbb{C}$ and we define $m : \mathbb{S}^1 \to \mathbb{S}^1$ by $m(\mathbf{z}) = \mathbf{z}^n$, then the composition $m \circ f^b : [0,1] \to \mathbb{S}^1$ has the property that $(m \circ f^b)(0) = (m \circ f^b)(1)$. By identifying the endpoints of $[0, 1]$, we obtain a map $g : \mathbb{S}^1 \to \mathbb{S}^1$, and the winding number of g is equal to the wrapping number of f^b.

It is not immediately clear why the wrapping number does not depend on our choice of representative, but it is not difficult to prove.

Lemma 7.1.5. *The wrapping number is well-defined.*

Proof. Let f be a representative of the $(b, \cdot)$-boundary braid β for which f^b lies in the boundary; we temporarily denote the wrapping number by $w_f(b, \cdot)$ to indicate the presumed dependence on our choice of representative. If f and f' both represent β, then $f' \cdot f^{-1}$ is a representative for the trivial braid with $w_{f' \cdot f^{-1}}(b, b) = w_{f'}(b, \cdot) - w_f(b, \cdot)$. It therefore suffices to show that each strand in every representative of the trivial braid has wrapping number zero.

Now, let f be a representative of the trivial braid e for which f^b lies in the boundary, and suppose that the wrapping number $w_f(b, b) \neq 0$. If $f^{b'}$ is another strand in f, then we may obtain a map to the pure braid group $\mathrm{PB{\small RAID}}_2$ by forgetting all strands except f^b and $f^{b'}$. The image β' of the trivial braid under this map may be written as an even power of δ_2 since ${\delta_2}^2$ generates $\mathrm{PB{\small RAID}}_2$, and the wrapping numbers can then be related as $w_e(b, b) = \frac{n}{2} w_{\beta'}(b, b)$. However, it is clear from the procedure of forgetting strands that the resulting braid in $\mathrm{PB{\small RAID}}_2$ is trivial, and since every braid in $\mathrm{PB{\small RAID}}_2$ has both strands boundary-parallel, we know that $w_{\beta'}(b, b)$ is zero, and thus so is $w_e(b, b)$. Therefore, every representative for the trivial braid has trivial wrapping numbers, and we are done. $\square$

As a consequence of Lemma 7.1.5, we are free to refer to the wrapping numbers of a *braid* rather than their representatives.

As one would intuitively hope, the wrapping number respects the composition of braids and it is zero precisely when a strand is fixed. The following lemmas may be quickly deduced from the definitions and Lemma 7.1.5.

Lemma 7.1.6. *If β and γ are braids in* $\textsc{Braid}_n$ *such that β is a (b,c)-boundary braid and γ is a (c,d)-boundary braid, then $\beta\gamma$ is a (b,d)-boundary braid with wrapping number $w_{\gamma\beta}(b,d) = w_\beta(b,c) + w_\gamma(c,d)$.*

Lemma 7.1.7. *Let $B \subseteq [n]$ and $\beta \in \textsc{Braid}_n(B,\cdot)$. Then $\beta \in \textsc{Fix}_n(B)$ if and only if $w_\beta(b,\cdot) = 0$ for all $b \in B$.*

Proof. If $\beta \in \textsc{Fix}_n(B)$, then there is a representative f of β in which each $(b,\cdot)$-strand is fixed and thus $w_\beta(b,\cdot) = 0$.

For the other direction, we begin with the case that $B = \{b\}$. Let f be a representative of $\beta \in \textsc{Braid}_n(B,\cdot)$ with the $(b,\cdot)$ strand in the boundary of D_n, and suppose that $w_\beta(b,\cdot) = 0$. Then the strand f^b begins and ends at the vertex p_b, and there is a homotopy $f(t)$ of f which moves every other strand off the boundary without changing f^b. That is, $f^{b'}(t) \notin \partial D_n$ whenever $b' \in [n] - \{b\}$ and $0 < t < 1$. After performing this homotopy, we note that $f(1)$ is a representative of β in which the (b,b)-strand has wrapping number 0 and there are no other braids in the boundary. Thus, there is a homotopy of this strand to the constant path, and therefore $\beta \in \textsc{Fix}_n(\{b\})$.

More generally, if $B \subseteq [n]$, then the set of braids $\beta \in \textsc{Braid}_n(B, \cdot)$ with $w_\beta(b, \cdot) = 0$ for all $b \in B$ are those which lie in the intersection of the fixed subgroups $\textsc{Fix}_n(\{b\})$. By Proposition 5.6.8, this is equal to $\textsc{Fix}_n(B)$ and we are done. $\square$

It straightforward to see that each braid β in $\textsc{Fix}_B(\mathrm{DS}_n(B, \cdot))$ has wrapping numbers $w_\beta(b, \cdot) = 0$ for each $b \in B$. It follows from Lemma 7.1.7 that the set of dual simple braids in $\textsc{Fix}_B(DS_n(B, \cdot))$ is precisely those with wrapping numbers equal to zero. Similarly, each braid in $\textsc{Move}_B(\mathrm{DS}_n(B, \cdot))$ has a simple description in terms of wrapping numbers.

Lemma 7.1.8. *Let $B \subseteq [n]$. If $\delta_\pi \in \textsc{Move}_B(\mathrm{DS}_n(B, \cdot))$, then $w_{\delta_\pi}(b, \cdot) \in \{0, 1\}$ for all $b \in B$.*

Proof. Follows immediately from Definition 5.4.11. $\square$

Lemma 7.1.7 gives us a simple condition for when a B-boundary braid has all its wrapping numbers equal to 0; the case when every wrapping number is 1 is also familiar.

Lemma 7.1.9. *Let $B \subseteq [n]$ and let $\hat{1}_B$ be the maximum element of $\textsc{Move}_B(\mathrm{DS}_n(B, \cdot))$. Then for all $b \in B$, the strand which begins at p_b ends at p_{b+1} and $w_{\hat{1}_B}(b, \cdot) = 1$.*

Proof. Recall from Remark 5.4.14 that if $B \subseteq [n]$, then the maximum element of the poset $\textsc{Move}_B(\mathrm{DS}_n(B, \cdot))$ is ${\delta_A}^{-1}\delta_n$, where $A = [n] - B$. Since the $(b, \cdot)$ strand of δ_A is boundary-parallel for each $b \in B$ by Lemma 7.1.2 and we have $w_{{\delta_A}^{-1}}(b, b) = 0$ and $w_{\delta_n}(b, b+1) = 1$ for all $b \in B$, the claim follows from Lemma 7.1.6. $\square$

For general braids, wrapping numbers may vary somewhat, but they cannot stray too far from each other.

Lemma 7.1.10. *Let $b_1, \ldots, b_k$ be integers satisfying $0 < b_1 < \cdots < b_k \leq n$ and suppose that $\beta \in \mathrm{B{\small RAID}}_n$ is a $(b_i, \cdot)$-boundary braid for every i. Then*

$$b_1 + w_\beta(b_1, \cdot) < b_2 + w_\beta(b_2, \cdot) < \cdots < b_k + w_\beta(b_k, \cdot) < b_1 + w_\beta(b_1, \cdot) + n.$$

Proof. Notice that it suffices to prove that

$$b_i + w_\beta(b_i, \cdot) < b_j + w_\beta(b_j, \cdot) < b_i + w_\beta(b_i, \cdot) + n$$

whenever $i < j$ or, in other words, that

$$w_\beta(b_j, \cdot) - w_\beta(b_i, \cdot) \in (b_i - b_j, b_i - b_j + n).$$

As a first case, suppose both $w_\beta(b_i, \cdot)$ and $w_\beta(b_j, \cdot)$ are divisible by n. Then forgetting all but the (b_i, b_i)- and (b_j, b_j)-strands of β yields a pure braid $\beta' \in \mathrm{PB{\small RAID}}_2$ which can be expressed as $\beta' = {\delta_2}^{2l}$ for some $l \in \mathbb{Z}$ since $\mathrm{PB{\small RAID}}_2 = \langle {\delta_2}^2 \rangle$. Then

$$w_\beta(b_h, b_h) = \frac{n}{2} w_{\beta'}(b_h, b_h) = nl$$

for each $h \in \{i, j\}$ and since every two-strand braid has simultaneously boundary-parallel strands with equal wrapping numbers, we conclude that $w_\beta(b_i, b_i) = w_\beta(b_j, b_j)$. Therefore, $w_\beta(b_j, b_j) - w_\beta(b_i, b_i) = 0$, which satisfies the inequalities above.

For the general case, define

$$\gamma = \beta {\delta_n}^{-w_\beta(b_i, \cdot)}$$

and observe that $w_\gamma(b_i, \cdot) = 0$. Notice that $w_\gamma(b_j, \cdot)$ is not congruent to $b_i - b_j$ mod n; if it were, then the $(b_i, \cdot)$- and $(b_j, \cdot)$-strands of γ would terminate in the same vertex. Let e

then be the representative of $w_\gamma(b_j,\cdot)$ mod n which lies in the interval (b_i-b_j, b_i-b_j+n). Then we can "unwind" each strand except the $(b_i,\cdot)$-strand via multiplication by $\delta_{[n]-\{b_i\}}$ to straighten out the $(b_j,\cdot)$-strand. That is, if we define

$$\alpha = \gamma {\delta_{[n]-\{b_i\}}}^{-e}$$

we can see that α has both its $(b_i,\cdot)$- and $(b_j,\cdot)$-strands boundary-parallel with wrapping numbers $w_\alpha(b_i,\cdot)=0$ and $w_\alpha(b_j,\cdot)\equiv 0 \pmod n$. By the first case above, we can promote equivalence mod n to an equality and we have $w_\alpha(b_j,\cdot)=0$ and thus $w_\gamma(b_j,\cdot)=e$. Then $w_\beta(b_j,\cdot)=e+w_\beta(b_i,\cdot)$ and finally,

$$w_\beta(b_j,\cdot)-w_\beta(b_i,\cdot)=e\in(b_i-b_j, b_i-b_j+n)$$

as desired. □

The inequalities given above are sharp; any proposed numbers satisfying the hypotheses for Lemma 7.1.10 can be realized as the wrapping numbers for a braid. To see this, we present the following technical lemmas.

Remark 7.1.11 (Notation). For a fixed positive integer n, if $b\in\mathbb{Z}$, then let $\bar{b}$ be the unique integer in $[n]$ equivalent to b mod n.

Lemma 7.1.12. *Let $b_1,\ldots b_k$ be integers such that $b_1<b_2<\cdots<b_k<b_1+n$. Then there is a braid $\beta\in\mathrm{B{\small RAID}}_n$ with representative f such that for each $i\in[k]$, the strand beginning at p_i ends at $p_{\bar{b}_i}$ and remains in the boundary, with corresponding wrapping number $w_\beta(i,\bar{b}_i)=b_i-i$.*

Proof. When $b_1 = 1$, the proof is straightforward since by Lemma 7.1.10, the assumed inequalities allow us to keep the $(1, 1)$-strand fixed and compose braids which move the other strands sequentially into the specified positions.

In the general case, let $b_1 < b_2 < \cdots < b_k < b_1 + n$. For each $i \in [k]$, define $d_i = b_i - b_1 + 1$. Then $d_1, \ldots, d_k$ satisfy the requirements of the case above, so let β be the braid with representative such that the $(i, \overline{d}_i)$ strand remains in the boundary and $w_\beta(i, \overline{d}_i) = d_i - i$. Since the natural representative of δ_n leaves all strands in the boundary, $\beta{\delta_n}^{b_1-1}$ has a representative with the $(i, \cdot)$-strand in the boundary. Each $(i, \cdot)$ strand in this representative terminates at $d_i + b_1 - 1 = b_i$ and by Lemma 7.1.6, the $(i, \overline{b}_i)$-strand has wrapping number $w_{\beta{\delta_n}^{b_1-1}}(i, \overline{b}_i) = b_i - i$. □

Lemma 7.1.13. *Let $b_1, \cdots, b_k$ and $c_1, \ldots, c_k$ be integers with $b_1 < b_2 < \cdots < b_k < b_1 + n$ and $c_1 < c_2 < \cdots < c_k < c_1 + n$. Then there is a braid $\beta \in \mathrm{BRAID}_n$ with representative f such that for each $i \in [k]$, the strand beginning at p_{b_i} ends at p_{c_i} and remains in the boundary with wrapping number $w_\beta(\overline{b}_i, \overline{c}_i) = c_i - b_i$.*

Proof. Let β_1 and β_2 be the braids provided by Lemma 7.1.12 which satisfy the inequalities $b_1 < b_2 < \cdots < b_k < b_1 + n$ and $c_1 < c_2 < \cdots < c_k < c_1 + n$. That is, β_1 and β_2 have representatives f_1 and f_2 such that the $(i, \overline{b}_i)$-strand of f_1 and the $(i, \overline{c}_i)$-strand of f_2 each lie in the boundary, with $w_{\beta_1}(i, \overline{b}_i) = b_i - i$ and $w_{\beta_2}(i, \overline{c}_i) = c_i - i$. Then ${\beta_1}^{-1}\beta_2$ is represented by ${f_1}^{-1} \cdot f_2$, for which the $(\overline{b}_i, \overline{c}_i)$-strand lies in the boundary with wrapping numbers $w_{{\beta_1}^{-1}\beta_2}(\overline{b}_i, \overline{c}_i) = c_i - b_i$. □

Corollary 7.1.14. *Let $b_1, \dots, b_k$ be integers satisfying $0 < b_1 < \cdots < b_k \leq n$ and suppose that there are integers $w_1, \dots, w_k$ with the property that*

$$b_1 + w_1 < b_2 + w_2 < \cdots < b_k + w_k < b_1 + w_1 + n.$$

Then there is a braid $\beta \in \textsc{Braid}_n$ such that β is a $(b_i, \cdot)$-boundary braid for each i and $w_i = w_\beta(b_i, \cdot)$.

Proof. Let $c_i = b_i + w_i$ and apply Lemma 7.1.13. $\square$

Combining Lemma 7.1.10 and Corollary 7.1.14 tells us that for each $B \subseteq [n]$, the braids with $(b, \cdot)$-strand boundary-parallel for each $b \in B$ can be described precisely by the integer lattice points in an intersection of half spaces prescribed by the given inequalities. As we will make precise in Section 7.3, this corresponds to an isometric embedding of a dilated column in the dual braid complex. In the meantime, we may use this characterization of boundary braids to prove the following theorem.

Theorem 7.1.15. *Let $\beta \in \textsc{Braid}_n$. All boundary-parallel strands of β are simultaneously boundary-parallel.*

Proof. Let $B \subseteq [n]$ and suppose $\beta \in \textsc{Braid}_n$ is a $(b, \cdot)$-boundary braid for each $b \in B$. If we write $B = \{b_1, \dots, b_k\}$ with $0 < b_1 < b_2 < \cdots < b_k \leq n$, then the wrapping numbers $w_\beta(b_i, \cdot)$ satisfy the inequalities given by Lemma 7.1.10, so by Corollary 7.1.14 we may fix a B-boundary braid $\gamma \in \textsc{Braid}_n$ with the same wrapping numbers as β. By Lemma 7.1.6, $\beta\gamma^{-1}$ is a (b, b)-boundary braid with $w_{\beta\gamma^{-1}}(b, b) = 0$ for each $b \in B$. Applying Lemma 7.1.7, we know that $\beta\gamma^{-1} \in \textsc{Fix}_n(B)$ and in particular, this braid has a representative which is B-boundary parallel. Since γ is a B-boundary braid, we therefore know that $\beta = (\beta\gamma^{-1})\gamma$ is a B-boundary braid, as desired. $\square$

The main result of the section, stated in analogy with Proposition 5.6.8, follows directly from Lemma 7.1.15.

Theorem 7.1.16. *Intersections of sets of boundary braids are sets of boundary braids. Concretely, if $B \subseteq [n]$, then*

$$\textsc{Braid}_n(B, \cdot) = \bigcap_{b \in B} \textsc{Braid}_n(\{b\}, \cdot)$$

and equivalently,

$$\textsc{Braid}_n(C \cup D, \cdot) = \textsc{Braid}_n(C, \cdot) \cap \textsc{Braid}_n(D, \cdot)$$

for any $C, D \subseteq [n]$.

7.2 Fixing and Moving

Our motivation for the remainder of the chapter is to understand the full subcomplex of the dual braid complex on the vertices labeled by braids in $\textsc{Braid}_n(B, \cdot)$ for a fixed $B \subseteq [n]$.

Remark 7.2.1 (Edges in $\textsc{UConf}_k(\Gamma_n, \boxslash)$). Let $B \subseteq [n]$ with $|B| = k$ and consider the unordered orthoscheme configuration space $\textsc{UConf}_k(\Gamma_n, \boxslash)$. The vertices of this configuration space are labeled by k-element subsets of $\mathbb{Z}/n\mathbb{Z}$, so B determines a vertex $\mathbf{v}_B$ of $\textsc{UConf}_k(\Gamma, \boxslash)$. The directed edges based at $\mathbf{v}_B$ end at vertices labeled by k-element subsets of $\mathbb{Z}/n\mathbb{Z}$ which are obtained by adding either 0 or 1 to each element of B. Suppose $B' \subseteq [n]$ represents such a subset of $\mathbb{Z}/n\mathbb{Z}$. There are many (B, B')-boundary braids in $\mathrm{DS}_n(B, \cdot)$, but only one in $\textsc{Move}_B(\mathrm{DS}_n(B, \cdot))$. To see this, observe by Theorem 5.4.16 that there is a natural copy of $\textsc{Fix}_B(\mathrm{DS}_n(B, \cdot))$ within $\mathrm{DS}_n(B, \cdot)$ for each element of $\textsc{Move}_B(\mathrm{DS}_n(B, \cdot))$, and each copy is precisely the set of (B, B')-boundary braids for each possible choice of B'. Furthermore, each of these copies intersects $\textsc{Move}_B(\mathrm{DS}_n(B, \cdot))$ in a unique element. Thus we we may label the edge from $\mathbf{v}_B$ to the vertex $\mathbf{v}_{B'}$ by a unique braid in $\textsc{Move}_B(\mathrm{DS}_n(B, \cdot))$. The picture is that each motion of k points in the directed n-cycle Γ_n corresponds to a braid which moves k strands in the boundary of D_n in a minimal way. Repeating this process for each vertex in $\mathrm{U}\textsc{Conf}_k(\Gamma_n, \boxslash)$ yields a labeling of its 1-skeleton by the specified dual simple braids. We further remark that each ordered triangle is labeled by a pair of edges as described in Remark 4.3.10, and the three edges of the triangle correspond to moving a point along one edge, moving a point along the other, and moving a point along each simultaneously. The corresponding picture for the dual simple braids is that of moving two strands separately or together - if β labels the long edge of the triangle and β_1, β_2 label edges so that β_2 begins where β_1 ends, then $\beta = \beta_1\beta_2$.

Definition 7.2.2 (Move subcomplex)**.** Fix $B \subseteq [n]$ with $|B| = k$. As in Remark 7.2.1, fix $\mathbf{v}_B$ to be the vertex of $\mathrm{U}\textsc{Conf}_k(\Gamma_n, \boxslash)$ which is determined by B. If $B = \{b_1, \ldots, b_k\}$ with $1 \leq b_1 < \cdots < b_k \leq n$, then by the proof of Theorem 4.3.13, the universal cover of $\mathrm{U}\textsc{Conf}_k(\Gamma_n, \boxslash)$ is identified with a (k, n)- dilated column in $\mathbb{R}^k$ with a covering map which sends the point $\widetilde{\mathbf{v}}_B = (b_k, \ldots, b_1)$ in $\mathbb{R}^k$ to $\mathbf{v}_B$. By Remark 7.2.1, each directed edge in the dilated column inherits a label from $\textsc{UConf}_k(\Gamma_n, \boxslash)$ which is determined by its start and end vertices. Then for each vertex $\widetilde{\mathbf{u}}$ in the (k,n)-dilated column, each (not necessarily directed) path from $\widetilde{\mathbf{v}}_B$ to $\widetilde{\mathbf{u}}$ determines an element of $\textsc{Braid}_n(B,\cdot)$ by taking the corresponding ordered product of edge labels and their inverses, noting that the product is a boundary braid by repeated application of Lemma 7.1.6. Since the dilated column is simply connected, we can label $\widetilde{\mathbf{v}}_B$ by the identity braid and each other vertex by the well-defined braid obtained by the following paths from $\widetilde{\mathbf{v}}_B$ to that vertex. This gives a map from the vertices of the (k,n)-dilated column to $\textsc{Braid}_n$, and we refer to the image as $\textsc{Move}_n(B,\cdot)$. Furthermore, the coordinates of each vertex (via the embedding of the dilated column into $\mathbb{R}^k$) correspond to the wrapping numbers of the associated braid, so we can see that the resulting braids are uniquely determined by their wrapping numbers, and in particular, the map to $\textsc{Braid}_n$ is injective. This extends to a simplicial map to the dual braid complex $\mathcal{D}_n$ and thus determines an isometric embedding of the (k,n)-dilated column into $\mathcal{D}_n$. The image of this embedding is referred to as the *move subcomplex* and is denoted $\textsc{Cplx}(\textsc{Move}_n(B,\cdot))$.

As an immediate consequence of Proposition 3.5.9 and Definition 7.2.2, we record the following.

Corollary 7.2.3. *Let $B \subseteq [n]$. Then $\textsc{Cplx}(\textsc{Move}_n(B))$ is a $\mathrm{CAT}(0)$ subcomplex of the dual braid complex.*

By construction, observe that the braids labeling vertices in the move subcomplex are uniquely determined by their wrapping numbers. Combining this with Lemma 7.1.10 provides another characterization of the move subcomplex as a (k, n)-dilated column. By identifying braids with the k-tuple recording their wrapping numbers, one can check that the braid produced in Definition 7.2.2 via a path between two vertices is precisely the braid constructed in the proof of Lemma 7.1.13.

Definition 7.2.4 (Moving B)**.** Let $B \subseteq [n]$. Since each element of $\textsc{Move}_n(B, \cdot)$ is uniquely determined by its wrapping numbers, then each $\beta \in \textsc{Braid}_n(B, \cdot)$ may be associated to a unique braid in $\textsc{Move}_n(B, \cdot)$ with the same wrapping numbers. This gives a map $\textsc{Move}_B : \textsc{Braid}_n(B, \cdot) \to \textsc{Move}_n(B, \cdot)$, although we remark that neither $\textsc{Braid}_n(B, \cdot)$ nor $\textsc{Move}_n(B, \cdot)$ is a subgroup of $\textsc{Braid}_n$. When β is a dual simple braid, its wrapping numbers are each 0 or 1, and since elements of $\textsc{Move}_n(B, \cdot)$ are determined by their wrapping numbers, we know that $\textsc{Move}_B(\beta)$ lies in the subposet $\textsc{Move}_B(\mathrm{DS}_n(B, \cdot))$. Hence, this definition matches Definition 5.4.11 when restricted to $\mathrm{DS}_n(B, \cdot)$.

Lemma 7.2.5. *If $\beta \in \textsc{Braid}_n(B, B')$ and $\delta_\pi \in \mathrm{DS}_n(B', \cdot)$, then*

$$\textsc{Move}_B(\beta)\textsc{Move}_{B'}(\delta_\pi) = \textsc{Move}_B(\beta\delta_\pi).$$

In other words, if β_1 and β_2 are (B, B')-boundary braids such that ${\beta_1}^{-1}\beta_2$ is a dual simple braid, then $\textsc{Move}_B(\beta_1)^{-1}\textsc{Move}_B(\beta_2) = \textsc{Move}_{B'}({\beta_1}^{-1}\beta_2)$ is a dual simple braid in $\textsc{Move}_{B'}(\mathrm{DS}_n(B', \cdot))$.

Proof. If β_1 and β_2 are elements of $\textsc{Braid}_n(B, B')$, then ${\beta_1}^{-1}\beta_2 \in \textsc{Braid}_n(B', B')$. Suppose that ${\beta_1}^{-1}\beta_2$ is a dual simple braid - then by definition it is an element of $\mathrm{DS}_n(B', B')$, and for each $b \in B'$, the wrapping number $w_{{\beta_1}^{-1}\beta_2}(b)$ is either 0 or 1. Since the elements of $\textsc{Move}_n(B, \cdot)$ are determined by their wrapping numbers and $\textsc{Move}_B(\beta_i)$ has the same wrapping numbers as β_i, we then know that the wrapping numbers of $\textsc{Move}_B(\beta_1)$ and $\textsc{Move}_B(\beta_2)$ each differ by 0 or 1. In other words, there is an edge in the move subcomplex from $\textsc{Move}_B(\beta_1)$ to $\textsc{Move}_B(\beta_2)$ which is labeled by a dual simple braid in $\textsc{Move}_{B'}(\mathrm{DS}_n(B', \cdot)$. Since $\textsc{Move}_B$ is trivial on this subposet, we're done. □

Our second map on $\textsc{Braid}_n(B, \cdot)$ yields a more easily recognizable structure.

Definition 7.2.6 (Fixing B)**.** Let $B \subseteq [n]$. For each $\beta \in \textsc{Braid}_n(B, \cdot)$, define $\textsc{Fix}_B(\beta)$ to be the product $\beta\textsc{Move}_B(\beta)^{-1}$. By Lemma 7.1.6, $w_{\textsc{Fix}_B(\beta)}(b) = 0$ for all $b \in B$, so by Lemma 7.1.7 this gives a map $\textsc{Fix}_B : \textsc{Braid}_n(B, \cdot) \to \textsc{Fix}_n(B)$. If we restrict this map to the poset $\textsc{Fix}_B(\mathrm{DS}_n(B, \cdot))$, we can see by Definition 7.2.4 that the resulting map matches the map $\textsc{Fix}_B$ given in Definition 5.4.10. When β is a (B, B)-boundary braid, we may also obtain $\textsc{Fix}_B(\beta)$ in the following manner. Select a representative f for β where f^b lies in the boundary of D_n for each $b \in B$ and perturb the strands so that each strand not in B lies off the boundary. Then we may replace each f^b strand with a fixed strand, and the result is a representative for $\textsc{Fix}_B(\beta)$.

Lemma 7.2.7. *If $\beta \in \textsc{Braid}_n(B, B')$ and $\delta_\pi \in \mathrm{DS}_n(B', \cdot)$, then*

$$\textsc{Fix}_B(\beta\delta_\pi) = \textsc{Fix}_B(\beta)\textsc{Fix}_{B'}(\delta_\pi)^{\textsc{Move}_B(\beta)},$$

where we recall that x^y denotes yxy^{-1}.

Proof. Applying Lemma 7.2.5 and Definition 7.2.6, we have the following sequence of equalities:

$$\begin{aligned}
\textsc{Fix}_B(\beta\delta_\pi) &= (\beta\delta_\pi)\textsc{Move}_B(\beta\delta_\pi)^{-1} \\
&= (\beta\delta_\pi)(\textsc{Move}_B(\beta)\textsc{Move}_{B'}(\delta_\pi))^{-1} \\
&= (\beta\delta_\pi)(\textsc{Move}_{B'}(\delta_\pi)^{-1}\textsc{Move}_B(\beta)^{-1} \\
&= \beta\textsc{Fix}_{B'}(\delta_\pi)\textsc{Move}_B(\beta)^{-1} \\
&= \beta\textsc{Move}_B(\beta)^{-1}\textsc{Move}_B(\beta)\textsc{Fix}_{B'}(\delta_\pi)\textsc{Move}_B(\beta)^{-1} \\
&= \textsc{Fix}_B(\beta)\textsc{Fix}_{B'}(\delta_\pi)^{\textsc{Move}_B(\beta)}
\end{aligned}$$

□

Finally, we record the following proposition.

Proposition 7.2.8. *Let $B \subseteq [n]$. If $\beta \in \textsc{Braid}_n(B, \cdot)$, then $\beta = \textsc{Fix}_B(\beta)\textsc{Move}_B(\beta)$.*

Proof. Follows immediately from Definition 7.2.6. □

7.3 The Boundary Subcomplex

Just as the subgroup $\textsc{Fix}_n(B)$ is contained in the larger set of boundary braids $\textsc{Braid}_n(B, \cdot)$, so too is the dual parabolic subcomplex $\textsc{Cplx}(\textsc{Fix}_n(B)) = \mathcal{D}_A$ (where $A = [n] - B$) contained within the subcomplex of the dual braid complex determined by $\textsc{Braid}_n(B, \cdot)$. While the former complex is of strictly smaller dimension than the dual braid complex, the latter is a full-dimensional subcomplex. In this section, we use the maps defined in Section 7.2 to exhibit this larger subcomplex as a direct product, the main theorem of this chapter. Aside from the articulation of interesting portions of the braid group and their corresponding complexes, the main result has potential implications for the curvature of the braid group.

Definition 7.3.1 (Boundary Subcomplexes)**.** Let $B \subseteq [n]$. Define a local order on $\textsc{Braid}_n(B, \cdot)$ by declaring that for each $\beta_1, \beta_2 \in \textsc{Braid}_n(B, \cdot)$, $\beta_1 \leq \beta_2$ if and only if there is a dual simple braid $\delta_\pi \in \mathrm{DS}_n$ such that $\beta_2 = \beta_1 \delta_\pi$. Notice that if β_1 is a (B, B')-boundary braid, this can only be satisfied if $\delta_\pi \in \mathrm{DS}_n(B', \cdot)$. The orthoscheme complex for this order is then naturally a subcomplex of the dual braid complex $\mathcal{D}_n$, and we refer to this as the *boundary subcomplex* associated to B. Denote this complex by $\textsc{Cplx}(\textsc{Braid}_n(B, \cdot))$. The restriction of this local order to $\textsc{Fix}_n(B)$ or $\textsc{Move}_n(B)$ yields subcomplexes $\textsc{Cplx}(\textsc{Fix}_n(B))$ and $\textsc{Cplx}(\textsc{Move}_n(B))$. The latter is already known from Definition 7.2.2 as the *move subcomplex*, and the former is the *dual parabolic subcomplex* $\mathcal{D}_A$, where $A = [n] - B$ - see Definition 6.5.1.

Remark 7.3.2 (Notation). In this section, we revert to the use of $\textsc{Cplx}(L)$ to denote the orthoscheme complex of L in lieu of the more compact notation $\mathcal{D}_n$ for the dual braid complex. Our goal in doing so is to emphasize that these subsets are all obtained by restricting the local order on $\textsc{Braid}_n$ and thus correspond naturally to containment of subcomplexes.

We are now prepared to state and prove the main theorem of this chapter.

Theorem 7.3.3. *Let $B \subseteq [n]$. Then the map*

$$\varphi : \textsc{Braid}_n(B, \cdot) \to \textsc{Fix}_n(B) \times \textsc{Move}_n(B, \cdot)$$

which sends β to the ordered pair $(\textsc{Fix}_B(\beta), \textsc{Move}_B(\beta))$ induces an isometry

$$\textsc{Cplx}(\textsc{Braid}_n(B, \cdot)) \cong \textsc{Cplx}(\textsc{Fix}_n(B)) \boxslash \textsc{Cplx}(\textsc{Move}_n(B, \cdot))$$

on the corresponding orthoscheme complexes.

Proof. It suffices to show that the local order on $\textsc{Braid}_n(B, \cdot)$ given in Definition 7.3.1 is isomorphic to the direct product of the induced orders on $\textsc{Fix}_n(B)$ and $\textsc{Move}_n(B, \cdot)$. To this end, we check that the map φ is an isomorphism of local orders.

If β is a (B, B')-boundary braid, then β labels a vertex in the boundary subcomplex $\textsc{Cplx}(\textsc{Braid}_n(B, \cdot))$. By Definition 7.3.1, the directed edges leaving this vertex are labeled by the elements of $\mathrm{DS}_n(B', \cdot)$, which by Theorem 5.4.16 is isomorphic to the direct product of posets

$$\textsc{Fix}_{B'}(\mathrm{DS}_n(B', \cdot)) \times \textsc{Move}_{B'}(\mathrm{DS}_n(B', \cdot)).$$

On the other hand, $\varphi(\beta) = (\textsc{Fix}_B(\beta), \textsc{Move}_B(\beta))$ labels a vertex in the ordered simplicial product $\textsc{Cplx}(\textsc{Fix}_n(B)) \boxslash \textsc{Cplx}(\textsc{Move}_n(B, \cdot))$, and by Definition 7.3.1, the directed edges leaving this vertex are labeled by the elements of the direct product

$$\textsc{Fix}_B(\mathrm{DS}_n(B, \cdot)) \times \textsc{Move}_{B'}(\mathrm{DS}_n(B', \cdot)).$$

All that remains is to show that φ induces isomorphisms on these posets.

By Lemma 7.2.5, $\textsc{Move}_B$ sends each edge labeled by $\delta_\pi \in \mathrm{DS}_n(B', \cdot)$ to an edge labeled by $\textsc{Move}_{B'}(\delta_\pi)$. Hence, the restriction of $\textsc{Move}_B$ to $\textsc{Move}_{B'}(\mathrm{DS}_n(B', \cdot))$ is the identity map.

By Lemma 7.2.7, $\textsc{Fix}_B$ sends each edge labeled by $\delta_\pi \in \mathrm{DS}_n(B', \cdot)$ to an edge labeled by the conjugate $\textsc{Fix}_{B'}(\delta_\pi)^{\textsc{Move}_B(\beta)}$. Restricting this map to $\textsc{Fix}_{B'}(\mathrm{DS}_n(B', \cdot))$ yields an isomorphism to $\textsc{Fix}_B(\mathrm{DS}_n(B, \cdot))$ defined as conjugation by $\textsc{Move}_B(\beta)$.

Therefore, the map $\varphi : \beta \mapsto (\textsc{Fix}_B(\beta), \textsc{Move}_B(\beta))$ induces an isomorphism on the posets which label the directed edges leaving β and $(\textsc{Fix}_B(\beta), \textsc{Move}_B(\beta))$ in their respective orthoscheme complexes. Thus, this map is an isomorphism of the defining local orders and hence provides an isometry between the two orthoscheme complexes. $\square$

As a consequence of the theorem above together with Corollary 7.2.3, we may conclude the following curvature result.

Corollary 7.3.4. *If $B \subseteq [n]$ with $k = |B|$, then $\textsc{Cplx}(\textsc{Braid}_n(B, \cdot))$ is* CAT(0) *whenever $\textsc{Cplx}(\textsc{Braid}_{n-k})$ is* CAT(0)*.*

As stated before, it is known that $\textsc{Cplx}(\textsc{Braid}_n)$ is CAT(0) when $n \leq 6$; it is feasible that the corollary above could provide the needed technology for an inductive proof of the general case.

7.4 Boundary Braids in the Noncrossing Partition Link

As described in Section 6.5, each dual parabolic subgroup $\textsc{Braid}_A \subseteq \textsc{Braid}_n$ corresponds to a subcomplex in each of the dual braid complex, the cross-section complex, and the noncrossing partition link. Expanding our perspective to the set $\textsc{Braid}_n(B,\cdot)$ of boundary braids, we again obtain three corresponding subcomplexes, and each may be described in terms of the dual parabolic subcomplexes.

By Theorem 7.3.3, the boundary subcomplex $\textsc{Cplx}(\textsc{Braid}_n(B,\cdot))$ splits as the direct product of $\textsc{Cplx}(\textsc{Braid}_A)$ and a dilated column, where we define $A = [n] - B$ as usual. Notice that for each $\beta \in \textsc{Braid}_n(B,\cdot)$, we also have $\beta{\delta_n}^k \in \textsc{Braid}_n(B,\cdot)$ for each $k \in \mathbb{Z}$; the boundary subcomplex is thus a union of standard columns in the dual braid complex. In other words, $\textsc{Cplx}(\textsc{Braid}_n(B,\cdot))$ may be written as a direct product of $\mathbb{R}$ and a subcomplex of the cross-section complex. The vertex links in the latter component have a familiar structure.

Definition 7.4.1. Let $B \subseteq [n]$. The projection of $\textsc{Cplx}(\textsc{Braid}_n(B,\cdot))$ to the cross-section complex contains the vertex labeled by $\langle \delta_n \rangle$, and the corresponding vertex link is the subcomplex of $\mathcal{L}_n$ corresponding to the maximal chains in $\mathrm{NC}_n(B,\cdot)$. By Proposition 5.4.7, this is the subcomplex of $\mathcal{L}_n$ labeled by maximal chains which include each positive half-twist which sends $\mathbf{v}_b$ to $\mathbf{v}_{b+1}$, for all $b \in B$. We call this the *B-boundary link* of $\mathcal{L}_n$ and refer to this subcomplex of the link complex as $\mathcal{L}_n(B)$.

Remark 7.4.2. Let $B \subseteq [n]$ and define $A = [n] - B$ When $B = \{s\}$ for some $s \in [n]$, then $\textsc{Cplx}(\textsc{Braid}_n(B,\cdot))$ is the direct product of $\textsc{Braid}_A$ and $\mathbb{R}$ and the B-boundary link is precisely the subcomplex $p(\mathcal{L}_A)$. For larger B, this is not the case.

Unlike $p(\mathcal{D}_A)$, the subcomplex of the cross-section complex which corresponds to the maximal dual parabolic $\textsc{Braid}_A$, the analogous projection for $\textsc{Cplx}(\textsc{Braid}_n(B,\cdot))$ does not have isometric vertex links. Different choices for B lead to different links, depending on how close the elements of B are in $[n]$. Using the direct product structure proven in Theorem 7.3.3, we have the following theorem.

Theorem 7.4.3. *Let $B \subseteq [n]$, define $A = [n] - B$, and let m be the number of elements $b \in B$ with the property that $b + 1 \notin B$, reduced mod n. Then $\mathcal{L}_n(B)$ is isometric to the spherical join of $\mathbb{S}^{m-1}$ and $\mathcal{L}_A$.*

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